Exercise Session 1

Yoneda Lemma: let $\mathcal{C}$ be a (locally small) category. Then the functor

$$\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$$
$$X \longmapsto h_X = \mathrm{Hom}_{\mathcal{C}}(-, X)$$

is fully faithful.

I.e., can view $\mathcal{C}$ as a full subcategory of $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$. In particular, for $X, Y \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})}(h_X, h_Y)$$

$= \{$ families of morphisms $(\varphi(Z) : h_X(Z) \to h_Y(Z))_{Z \in \mathcal{C}}$ s.t. $\forall Z \to Z' \in \mathcal{C}$ the diagram

$$\begin{tikzcd} h_X(Z) \arrow[r, "\varphi(Z)"] & h_Y(Z) \\ h_X(Z') \arrow[u] \arrow[r, "\varphi(Z')"] & h_Y(Z') \arrow[u] \end{tikzcd}$$

commutes $\}$

In our case put $\mathcal{C} = \mathrm{Sch}/k$. Then $\mathrm{GrpSch}/k \subseteq \mathrm{Fun}(\mathrm{Sch}/k^{op}, \mathrm{Set})$. E.g., given $G \in \mathrm{GrpSch}/k$, can construct:

- $e : \mathrm{Spec}\, k \longrightarrow G$: Via Yoneda, e corresponds to a natural map

$$\begin{array}{ccc} h_{\mathrm{Spec}\,k}(S) & \longrightarrow & h_G(S) \\ \shortparallel & & \shortparallel \\ \{*\} & \longrightarrow & G(S) \end{array}$$

Just take the map $* \longmapsto e_{G(S)}$.

Easy to see that this is natural in S (i.e. the above square commutes $\forall S \to S'$)

- $i: G \to G$: Corresponds to $G(S) \to G(S)$, $g \mapsto g^{-1}$
- $c: G \times_k G \to G$, $(g,h) \mapsto hgh^{-1}$ (Note that $(G \times_k G)(S) = G(S) \times G(S)$)

<u>Def</u>: A group homomorphism of grp sch G, H over k is a natural transformation $\left(G(S) \xrightarrow{\varphi(S)} H(S)\right)_S$ s.t. each $\varphi(S)$ is a group hom.

$\rightsquigarrow$ Automatically induced by a map $G \to H$ of schemes over k.

<u>Examples</u>: • A group hom $\varphi: \mathbb{G}_m \to \mathbb{G}_m$ is given by a k-alg hom $\varphi^*: k[T^{\pm 1}] \to k[T^{\pm 1}]$ s.t. the diagram

$$\begin{tikzcd}
k[T^{\pm 1}] \otimes_k k[T^{\pm 1}] & k[T^{\pm 1}] \otimes_k k[T^{\pm 1}] \arrow[l, "\varphi^* \otimes \varphi^*"'] \\
k[T^{\pm 1}] \arrow[u, "m^*"] & k[T^{\pm 1}] \arrow[l, "\varphi^*"'] \arrow[u, "m^*"']
\end{tikzcd}$$

commutes. φ^* is determined by $\varphi^*(T) =: f \in k[T^{\pm 1}]$. Write $f = \sum_{k=-n_1}^{n_2} a_k T^k$. From the diagram we get

$$\sum_{k=-n_1}^{n_2} a_k (T \otimes 1)^k (1 \otimes T)^k = \left(\sum_{k=-n_1}^{n_2} a_k (T \otimes 1)^k\right) \cdot \left(\sum_{k=-n_1}^{n_2} a_k (1 \otimes T)^k\right)$$

$\Rightarrow f = T^k$ for some k.

$\rightsquigarrow \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \{[n]\}_{n \in \mathbb{Z}} \cong \mathbb{Z}$.

- $\mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m^\ell) = \{\text{natural grp homs } (\mathbb{G}_m(S)^n \to \mathbb{G}_m(S)^\ell)_S\}$

Yoneda! $= \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)^{n \cdot \ell}$

$\cong \mathrm{Mat}_{n \times \ell}(\mathbb{Z})$.

- $\mathbb{G}_a : Sch/k \to Grp$, $S \mapsto (\mathcal{O}_S(S), +)$ is represented by $\mathbb{G}_a = \operatorname{Spec} k[T]$.

  $m : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ is given by $T \mapsto T \otimes 1 + 1 \otimes T$

- $GL_n = \operatorname{Spec}\left(k[T_{ij}]_{i,j=1}^{n}[S] / (\det(T_{ij}) \cdot S - 1)\right)$

$$m^* : T_{ij} \longmapsto \sum_{k=1}^{n} T_{ik} \otimes T_{kj}$$

$$S \longmapsto S \otimes S$$

<u>Def:</u> Let $\varphi : G \to H$ hom of grp sch/k. Then

$$\ker \varphi := G \times_H \operatorname{Spec} k, \quad \text{where } \operatorname{Spec} k \to H \text{ is } e_H.$$

universal property of fiber product

<u>Note:</u> • $(\ker \varphi)(S) = G(S) \times_{H(S)} \underbrace{\operatorname{Spec} k(S)}_{= \{*\}} = \ker(\varphi(S) : G(S) \to H(S))$

$\rightsquigarrow$ Automatically get group scheme structure on $\ker \varphi$

• $e_H : \operatorname{Spec} k \to H$ closed immersion $\Rightarrow$ $\ker \varphi \subseteq G$ closed subscheme.

<u>Lemma:</u> Every grp sch G/k is separated.

Analogous to: G top. grp s.t. $\{e\} \subseteq G$ closed $\Rightarrow G$ Hausdorff

<u>Proof:</u>
$$\begin{tikzcd} G \arrow[r, "\Delta_G"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & G \times_k G \arrow[d, "m \circ (id \times i)"] \\ \operatorname{Spec} k \arrow[r, "e"] & G \end{tikzcd}$$
is cartesian, hence e closed immersion $\Rightarrow \Delta_G$ closed immersion. $\square$

<u>Examples:</u> • $\mu_n = \ker(\mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m) = \operatorname{Spec}(k[T^{\pm 1}] / (T^n - 1))$.

If $k = \bar{k}$ and $n \in k^\times$ then $T^n - 1 = (T - \zeta_n) \cdots (T - \zeta_n^n)$ for prim. n-th root of unity ζ_n

$$\Rightarrow \mu_n = \operatorname{Spec}\left(\prod_{k=1}^{n} k[T^{\pm 1}]/(T-\zeta_n^k)\right) \cong \operatorname{Spec}\left(\prod_{k=1}^{n} k\right) = \coprod_{k=1}^{n} \operatorname{Spec} k$$

$$\Rightarrow \mu_n = (\mathbb{Z}/n\mathbb{Z})_k.$$

If $n = p = \operatorname{char} k$ then $\mu_p = \operatorname{Spec} k[T]/(T-1)^p$ not reduced!

- Let $\operatorname{char} k = p$. Then $\mathrm{Fr}: \mathbb{G}_a \to \mathbb{G}_a$, $g \mapsto g^p$ is grp hom

$$\rightsquigarrow \alpha_p := \ker \mathrm{Fr} = \operatorname{Spec} k[T]/(T^p)$$

Note that $\alpha_p \cong \mu_p$ as schemes, but not as grp schemes.