Warning: The difficulty of the following exercises is not uniform: some of these just involve checking definitions while others are more substantial. Some of these exercises may also require you to look up definitions which were not covered in class. Unless otherwise indicated, we always follow the conventions and notation from class (e.g. manifolds are smooth, vector spaces are finite dimensional, etc.)

1. Let (M, ω) be a symplectic manifold.

   (i) Show by example that a submanifold of M need not be symplectic. Give examples of symplectic and Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_0)$.

   (ii) Let J be an almost-complex structure which is compatible with ω. Suppose that $V \subset M$ is a submanifold with the property that $TV \subset TM|_V$ is preserved by J. Then $(V, \omega|_V) \subset (M, \omega)$ is a symplectic submanifold and $J|_V$ is compatible with $\omega|_V$. (In particular, smooth complex submanifolds of complex projective space are symplectic with respect to the restriction of the Fubini–Study form. More generally, complex submanifolds of Kähler manifolds are Kähler.)

2. Let M be a closed manifold endowed with a volume form. Prove that the group of orientation-preserving diffeomorphisms deformation retracts onto its subgroup of volume-preserving diffeomorphisms. *Hint: Moser's method.*

3. Consider a family of smooth closed complex submanifolds of $\mathbb{CP}^n$ parametrized by a manifold B. By definition, this is just a submanifold $X \subset B \times \mathbb{CP}^n$ such that the composition $X \hookrightarrow B \times \mathbb{CP}^n \xrightarrow{\pi_1} B$ is a submersion and the fibers $X_b \subset \mathbb{CP}^n$ are closed complex submanifolds. Letting ω denote the standard Fubini–Study form on $\mathbb{CP}^n$, prove that the fibers $(X_b, \omega|_{X_b})$ are symplectomorphic. *Hint: we did a special case of this statement in class.*

4. Let M be a manifold. Convince yourself of the following statements:

   (i) An almost complex structure on a $2n$-dimensional manifold M is the same thing (up to homotopy) as a lift of the classifying map $M \to BGL(2n; \mathbb{R})$ to $BGL(n, \mathbb{C})$.

   (ii) A non-degenerate 2-form $\omega \in \Omega^2(M)$ is the same thing (up to homotopy) as a lift of the classifying map $M \to BGL(2n; \mathbb{R})$ to $BSp(2n, \mathbb{R})$.

(iii) The data of a non-degenerate 2-form ω and a compatible almost-complex structure J is the same thing (up to homotopy) as a lift of the classifying map $M \to BGL(2n;\mathbb{R})$ to $BU(n,\mathbb{R})$.

Remark. In class, we saw that any symplectic manifold admits a compatible almost-complex structure, and moreover the space of these is contractible. In view of the above exercise, this amounts to the observation that $Sp(2n,\mathbb{R})$ deformation retracts onto its maximal compact subgroup $BU(n,\mathbb{R})$.

5. Prove the following statements from class.

   (i) The map

$$\begin{aligned} PSL(2,\mathbb{C}) &\to Aut(\mathbb{CP}^1) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto (z \mapsto \frac{az+b}{cz+d}) \end{aligned}$$

   is an isomorphism of groups.

   (ii) $Aut(\mathbb{CP}^1)$ is 3-transitive.