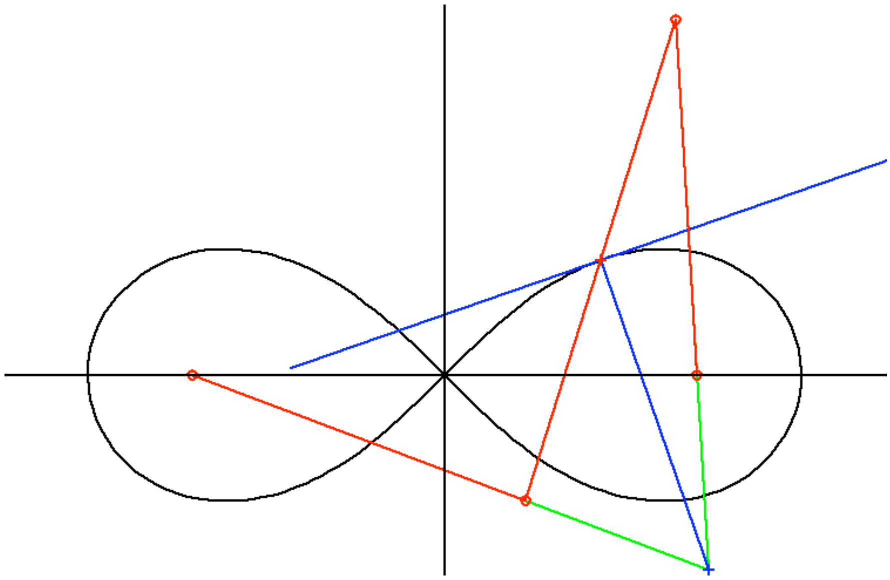
# Lemniscate *



The Lemniscate is a figure-eight curve with a simple **mechanical construction** attributed to Bernoulli: Choose two 'focal' points $F_1, F_2$ at distance $L := 2 * dd$, then take three rods, one of length $L$, two of length $R = L/\sqrt{2}$. The short ones can rotate around the focal points and the long one connects their free ends with rotating joints (red lines in the figure). This machine has one degree of freedom and the *midpoint* of the long rod traces out the Lemniscate while the short rods rotate (not uniformly). – This drawing mechanism will also work for arbitrary lengths $0 < R < L, R := cc$. The default morph in 3DXM varies $cc$. Another interesting morph is obtained by varying the position of the drawing pen on the long rod with $ff \in (0, 1)$. Click the `Init To Current Parameters` button in `Set Morphing`, then put $f0 := 0,\ f1 := 1$.

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

Since the Bernoulli Lemniscate is much better known than curves drawn by mechanisms with parameters different from $L : R = \sqrt{2} : 1$, we will give, below, parametrizations and equations only for the Bernoulli curve.
The curves in 3DXM are obtained as follows:
The endpoint of the right rod rotates with constant speed, i.e. $P(t) = (dd + cc \cdot \cos(t), cc \cdot \sin(t))$. The endpoint $Q(t)$ of the left rod is obtained by intersecting two circles (of radius $R$ around $F_1$ and radius $L$ around $P(t)$). One of the intersection points is $P'(t) = (-dd + cc \cdot \cos(t),\ cc \cdot \sin(t))$, since $|F_2 - F_1| = |P(t) - P'(t)| = L = 2 \cdot dd$. Therefore $Q(t)$ is obtained by reflecting $P'(t)$ in the Diagonal $\overline{F_1 P(t)}$ of the parallelogram $F_2, F_1, P'(t), P(t)$.
The drawing pen is at $ff \cdot P(t) + (1 - ff) \cdot Q(t)$.

*Mechanical constructions* of curves give rise to simple **tangent constructions**. We imagine that a plane is attached to the long rod. Then every point of this plane traces out a curve when the rods move. The velocity vectors of these traced curves give, at each moment, a vectorfield, that has concentric circles as integral curves (or, exceptionally, parallel lines). The centers of these concentric circles are the *momentary centers of rotation* for the moving plane. If we join a point of a traced curve to the corresponding momentary center of rotation, then this *radius* (drawn blue) is orthogonal to the tangent (also blue).

How can one find the momentary center of rotation for the current drawing machine? The endpoints of the two

short rods are points of the moving plane. We know that each can only move orthogonally to its rod (namely rotate around the other endpoint, a focal point). This says that both short rods point to the momentary center of rotation, which therefore is obtained as the intersection of two lines (drawn green in the figure).
Compare the other mechanically constructed curves.

Parametrizations are not unique, here is a well known one:

$$x(t) := \cos(t)/(1+\sin(t)^2)$$
$$y(t) := \sin(t)\cdot\cos(t)/(1+\sin(t)^2).$$

The Bernoulli Lemniscate has this implicit equation:

$$(x^2+y^2)^2 = x^2 - y^2.$$

Divide this by $r^2 := x^2+y^2$ to get the polar form:

$$r^2 = \cos(\phi)^2 - \sin(\phi)^2.$$

The points $F_1, F_2 := \pm 1/\sqrt{2}$ are called Focal points of the Lemniscate because of the special property:

$$|P - F_1| \cdot |P - F_2| = |F_1 - F_2|^2/4.$$

If one takes the complex square root of a circle which touches the $y$-axis from the right at 0 then one also obtains (half of) a Lemniscate. In the Conformal Category, choose $z \to \sqrt{z}$, and then in the `Action Menu`, select `Choose Circle by Mouse`, and create a circle that is tangent to the $y$-axis at 0.

The inversion map: $(x,y) \mapsto (x,y)/(x^2+y^2)$ often transforms some interesting curve into another interesting curve. And indeed, the Lemniscate, with the above parametrization, is transformed by inversion into the curve

$$x = 1/\cos(t),\ y = \sin(t)/\cos(t).$$

Observe the implicit equation $x^2 - y^2 = 1$. It shows that the new curve is a hyperbola with orthogonal asymptotes. So we could have obtained the Bernoulli Lemniscate from the orthogonal hyperbola by inversion in a circle around its midpoint. – More generally, inversions of hyperbolae $x^2/a^2 - y^2/b^2 = const$ give figure 8 curves with non-orthogonal double tangents. The angle $2\alpha$ between the double tangents is the same as the angle between the asymptotes and satisfies $\tan\alpha = b/a$. The angle $2\beta$ between the double tangents of the figure 8 curves of our drawing mechanisms satisfies $\sin\beta = R/L$. Set $a := R,\ b := a/\sqrt{L^2/R^2 - 1}$ to obtain $\alpha = \beta$. Invert, with $\vec{x} \mapsto \vec{x} \cdot a^2/|\vec{x}|^2$, the lemniscate and put the result into the term $x^2/a^2 - y^2/b^2$ to find that it is $\mathbf{1}$. This gives implicit lemniscate equations: $x^2/a^2 - y^2/b^2 = (x^2 + y^2)^2/a^4$. And, invert hyperbola parametrizations, e.g. $x = a/\cos(t)$, $y = b\sin(t)/\cos(t)$, to parametrize lemniscates.

We note that not every figure 8 curve (with orthogonal double tangents) is a Bernoulli Lemniscate. Another figure-eight is obtained by the simpler parametrization:

$$x(t) := \cos(t), \quad y(t) := \sin(t) \cdot \cos(t),$$

which has the implicit equation $y^2 = x^2(1 - x^2)$.

H.K.