Conchoid of Nicomedes *

History

According to common modern accounts, the conchoid of Nicomedes was first conceived around 200 B.C by Nicomedes, to solve the angle trisection problem. The name conchoid is derived from Greek meaning "shell", as in the word conch. The curve is also known as cochloid.

From E. H. Lockwood (1961):

> The invention of the conchoid ('mussel-shell shaped') is ascribed to Nicomedes (second century B.C.) by Pappus and other classical authors; it was a favourite with the mathematicians of the seventeenth century as a specimen for the new method of analytical geometry and calculus. It could be used (as was the purpose of its invention) to solve the two problems of doubling the cube and of trisecting an angle; and hence for every cubic or quartic problem. For this reason, Newton suggested that it should be treated as a 'standard' curve.

*This file is from the 3D-XploreMath project.
Please see http://rsp.math.brandeis.edu/3D-XplorMath/index.html

Description

The Conchoid of Nicomedes is a one parameter family of curves. They are special cases of a more general conchoid construction, being the conchoids of a line.

Step-by-step explanation:

1. Given a line ℓ, a point O not on ℓ,, and a distance k.

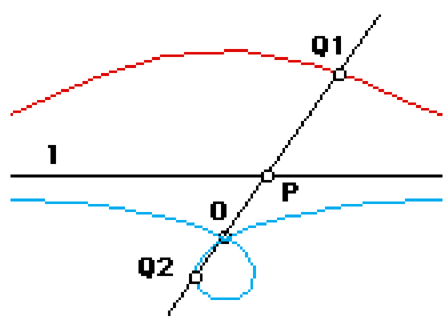
2. Draw a line m passing through O and any point P on ℓ.

3. Mark points $Q1$ and $Q2$ on m such that

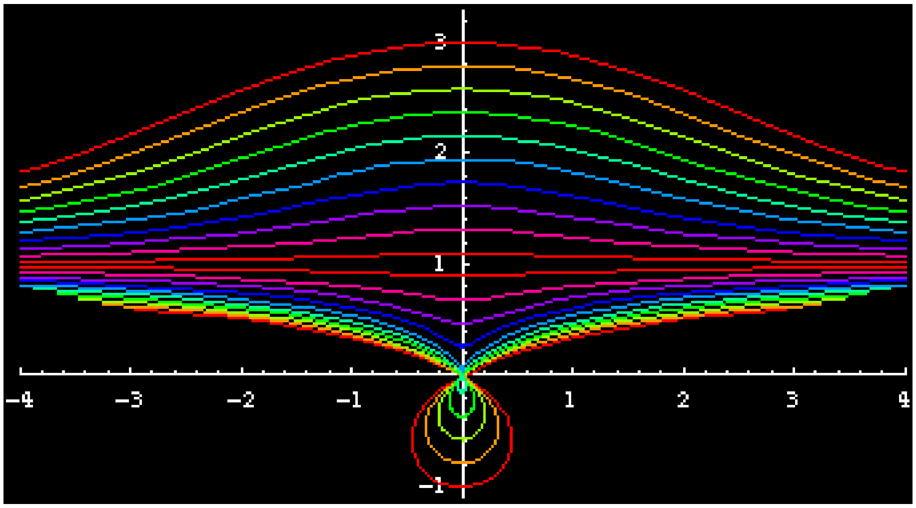
$$distance[Q1, P] = distance[Q2, p] = k.$$

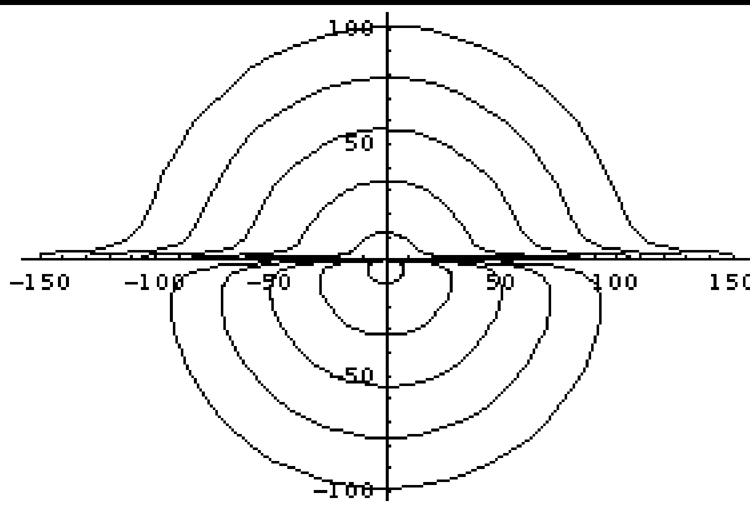
4. The locus of $Q1$ and $Q2$ as P varies on ℓ is the conchoid of Nicomedes.

The point O is called the *pole* of the conchoid, and the line ℓ is called its *directrix*. It is an asymptote of the curve.



The following figures shows the curve family. The pole is taken to be at the origin, and directrix is $y = 1$. The figure on top has constants k from -2 to 2. The one below has constants k from -100 to 100.





Formulas

Let the distance between pole and line be b, and the given constant be k. The curve has only the one parameter k, because for a given b, all families of the curve can be generated by varying k (they differ only in scale). (Similarly, we could use b as the parameter.) In a mathematical context, we should just use $b = 1$, however, it is convenient to have formulas that have both b and k. Also, for a given k, the curve has two branches. In a mathematical context, it would be better to define the curve with a signed constant k corresponding to a curve of only one branch. We will be using this intepretation of k. In this respect, the conchoid of Nichomedes is then two conchoids of a line with constants k and $-k$.

The curve with negative offset can be classified into three types: if $b < k$ there is a loop; if $b = k$, a cusp; and if $b > k$, it is smoothly imbedded. Curves with positive offsets are always smooth.

The following are the formulas for a conchoid of a line $y = b$, with pole O at the origin, and offset k.

Polar: $r = b/\sin(\theta) + k$, $-\pi/2 < \theta < \pi/2$.

This equation is easily derived: the line $x = b$ in polar equation is $r = b/\cos\theta$, therefore the polar equation is $r = b/\cos(\theta) + k$ with $-\pi/2 < \theta < \pi/2$ for a signed k (i.e., describing one branch.). Properties of cosine show that as θ goes from 0 to 2π, two conchoids with offset $\pm|k|$ results from a single equation $r = b/\cos(\theta) + k$. To rotate the graph by $\pi/2$, we replace cosine by sine.

Parametric: $(t + (kt)/\sqrt{b^2 + t^2}, b + (bk)/\sqrt{b^2 + t^2})$, $-\infty < t < \infty$.

If we replace t in the above parametric equation by $b\tan(t)$, we get the form: $(k + b/\cos(t))(\sin(t), \cos(t))$, $-\pi/2 < t < 3/2\pi$. $t \neq \pi/2$.

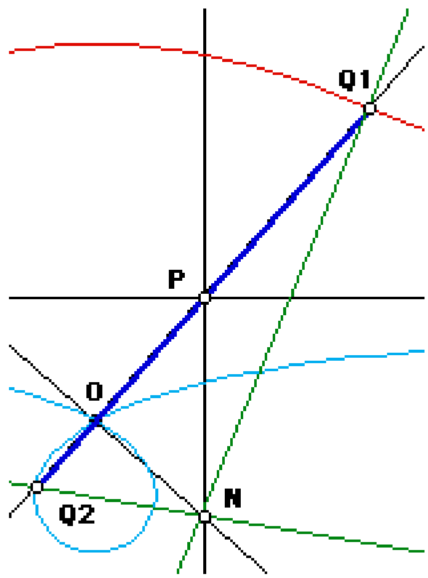
For conchoids of a line with positive and negative offsets k and pole at the origin, we have the

Implicit Cartesian equation: $(x^2 + y^2)(y - b)^2 = k^2 y^2$.
If $k < b$, the point at the origin is an isolated point.

If $k < 0$ and $b < |k|$, the conchoid has a loop with area $(b\sqrt{k^2 - b^2} - 2bk\ln((k + \sqrt{k^2 - b^2})/b) + k^2 \arccos(b/k))$. The area between any conchoid of a line and its asymptote is infinite.

Tangent Construction

Look at the conchoid tracing as a mechanical device, where a bar line $[O, P]$ slides on a line at P and a fixed joint O. The point P on the bar moves along the directrix, and the point at O moves in the direction of vector $[O, P]$. We know the direction of motion of the bar at the two joints O and P at any time. The intersection of normals to these directions form the instaneous center of rotation N. Since the tracing points $Q1$ and $Q2$ are parts of the apparatus, N is also their center of rotation and therefore line $[N, Q1]$ and line $[N, Q2]$ are the curve's normals.



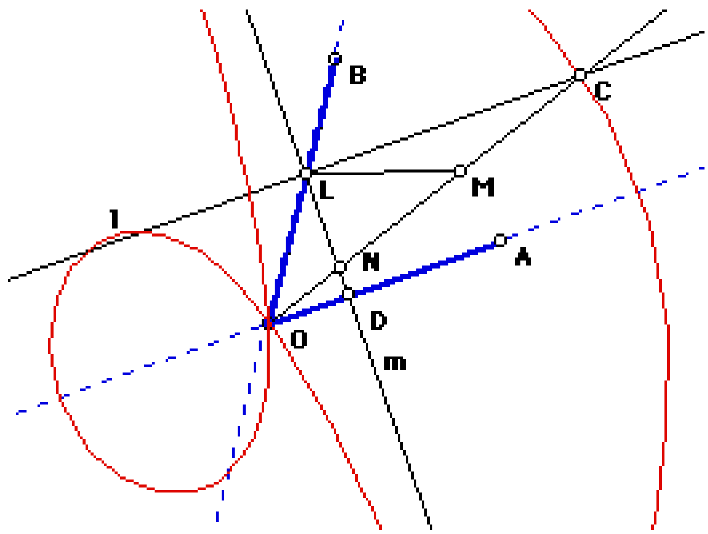
Angle Trisection

The curve can be used to solve the Greek Angle Trisection problem. Given an acute angle AOB, we want to construct an angle that is $1/3$ of AOB, with the help of conchoid of Nicomedes.

Steps: Draw a line m intersecting segment $[A, O]$ and perpendicular to it. Let D be intersection of m and the line $[A, O]$, L the intersection of m and the line $[B, O]$. Suppose we are given a conchoid of

Nicomedes, with pole at O, directrix m, and offset $2\,distance[O, L]$. Draw a line ℓ intersecting L and perpendicular to m. Let C be an intersection of the curve and ℓ, the one on the opposite side of the pole.

Theorem. $angle[A, O, B] = 3 * angle[A, O, C]$.



Proof. $angle[A, O, C] = angle[O, C, L]$ because the line $[O, C]$ cuts parallel lines. Let q be the line $[O, C]$, $N := \ell \cap [m, q]$, M the midpoint of $[N, C]$, and $k := distance[O, L]$.

By our construction, $distance[N, M] = distance[M, C] = k$. Since NLC is a right triangle, we see that MN, ML, MC, and OL all have the same length, thus triangle $[M, L, C]$ and triangle $[M, L, N]$ are isosceles, and it follows that $angle[N, M, L] = 2\,angle[M, C, L]$. Since $distance[O, L] = distance[M, L]$, triangle $[M, L, O]$ is also isosceles, and thus its two base angles are equal. This shows that an angle equal to angle $[A, O, C]$ is $1/3$ an angle equal to angle$[A, O, B]$. $\square$

The essential point where the conchoid makes the trisection possible is in the construction of the point C on ℓ such that $distance[N, C] = 2\, distance[O, L]$, where N is the intersection of m and the line $[O, C]$. Note that for each new angle to trisect, a new conchoid is needed. This is in contrast to some other trisectrixes such as the quadratrix, where all angles can be trisected once the curve is given.

The conchoid can also be used to solve the classic problem of doubling the cube.

XL.