Submersions via Projections

H. Karcher (Bonn). Geometria Dedicata 74, Nr.3 (1999), 249-260

Abstract. By writing the O'Neill tensors as derivatives of the natural projection one does not need the usual case distinctions any more and gets a much shorter list of basic equations. A short, induction free proof of the Frobenius theorem is a by-product.

Basic setup. Readers are invited to modify the assumptions according to their taste. Initially, the important data are two complementary subbundles HM, VM of the tangent bundle $TM = HM \oplus VM$ of a manifold M; with these we have two complementary projections $\mathcal{H}, \mathcal{V}$ to the subbundles, i.e. with $\mathcal{H} + \mathcal{V} = id$, $\mathcal{H} \cdot \mathcal{V} = 0 = \mathcal{V} \cdot \mathcal{H}$. Also, we are given a connection ∇ on M. Starting with our section on submersions a pseudo-Riemannian metric g with Levi-Civita connection ∇ on M and an integrable subbundle VM are naturally part of the data. Then, of course, one natural choice for $\mathcal{H}, \mathcal{V}$ are the orthogonal projections to the subbundle VM and to its orthogonal subbundle HM. But orthogonality is not always wanted: The two transversal families of lines on the one-sheeted hyperboloid of revolution give natural complementary projections which are not orthogonal, neither when considered in R^3 nor as a pseudo-Riemannian sphere. Any covariant derivative ∇ on TM is the sum of its HM- and VM-components, and one gets (as usual) the induced connections ∇^H, ∇^V for the subbundles. In the most general situation everything is symmetric in the subbundles HM, VM, but as more assumptions are added one subbundle gets preferred (e.g. becomes integrable); also, I prefer to elimate $\nabla\mathcal{V}$ using $\nabla\mathcal{V} = -\nabla\mathcal{H}$ (from $\mathcal{H} + \mathcal{V} = id$), even though that breaks the symmetry immediately. – The organisation of the paper is as follows: First I give the new version of the splitting of the connection in its VM and HM components, compare with the original definitions of O'Neill and Gray and show how basic symmetries come out more clearly. Next I define geodesics and the corresponding exponential map so that the case of nonintegrable subbundles is included; as application I prove Frobenius' theorem. Our main simplification lies in the curvature computations, the submersion assumption is only used for the horizontal curvatures. Finally, the family of Levi-Civita connections for the standard shrinking of the fibres fits well into this context.
I thank R.S. Palais, W. Ballmann and L. Berard Bergery for questions, objections and suggestions which improved this text.
Let X, Y, Z denote sections of TM, U, V, W denote sections in VM and H, K, L sections in HM. We will see that it is a major advantage to write the splitting of ∇ in terms of the *derivative* $\nabla\mathcal{H}$ by using the following product rule $\nabla_X(\mathcal{H} \cdot Y) = \nabla_X\mathcal{H} \cdot Y + \mathcal{H} \cdot \nabla_X Y$. We get

$$(1_H) \qquad \mathcal{H} \cdot \nabla_X V = -\nabla_X\mathcal{H} \cdot V = +\nabla_X\mathcal{V} \cdot V$$

$$(1_V) \qquad \mathcal{V} \cdot \nabla_X H = -\nabla_X\mathcal{V} \cdot H = +\nabla_X\mathcal{H} \cdot H,$$

hence the desired splitting of ∇ :

$$(2_H) \qquad \nabla_X H = \nabla_X^H H + \nabla_X\mathcal{H} \cdot H$$

$$(2_V) \qquad \nabla_X V = \nabla_X^V V - \nabla_X\mathcal{H} \cdot V.$$

Remark. The connection with the O'Neill tensors is:

$$
\begin{array}{lclcll}
A_K H & := & \mathcal{V}\cdot\nabla_K H & = & \nabla_K\mathcal{H}\cdot H & (= -\nabla_H\mathcal{H}\cdot K, \text{in case of submersions})\\
T_V H & := & \mathcal{V}\cdot\nabla_V H & = & \nabla_V\mathcal{H}\cdot H & \\
A_K V & := & \mathcal{H}\cdot\nabla_K V & = & -\nabla_K\mathcal{H}\cdot V & \\
T_U V & := & \mathcal{H}\cdot\nabla_U V & = & -\nabla_U\mathcal{H}\cdot V & (= -\nabla_V\mathcal{H}\cdot U, \text{ for integrable } VM).
\end{array}
$$

This can be expressed by the following tensor identities: $A = (\nabla_{\mathcal{H}.}\mathcal{H})\cdot(\mathcal{H}-\mathcal{V})$, $T = (\nabla_{\mathcal{V}.}\mathcal{H})\cdot(\mathcal{H}-\mathcal{V})$. Clearly, the derivatives ∇A, ∇T are such a mixture of first and second derivatives of $\mathcal{H}$, that the following symmetries are obscured for the O'Neill tensors.

Symmetries. There are immediate consequences of $\mathcal{H},\mathcal{V}$ being complementary projections. For (3_1) we assume the presence of a metric g (not necessarily a submersion metric), then the symmetry of the orthogonal projections $\mathcal{H},\mathcal{V}$ implies the symmetries of their derivatives, i.e. for all $X, X', Y, Z \in TM$ holds

$$
(3_1)\qquad
\begin{aligned}
g(\mathcal{H}\cdot Y, Z) &= g(Y, \mathcal{H}\cdot Z)\\
g(\nabla_X\mathcal{H}\cdot Y, Z) &= g(Y, \nabla_X\mathcal{H}\cdot Z)\\
g(\nabla^2_{X,X'}\mathcal{H}\cdot Y, Z) &= g(Y, \nabla^2_{X,X'}\mathcal{H}\cdot Z).
\end{aligned}
$$

Differentiating $\mathcal{H}\cdot\mathcal{V} = 0 = \mathcal{V}\cdot\mathcal{H}$ gives (using $\mathcal{H}+\mathcal{V} = id$, hence $\nabla\mathcal{H} = -\nabla\mathcal{V}$ again):

$$
(3_2)\qquad \nabla_Y\mathcal{H}\cdot\mathcal{V} = \mathcal{H}\cdot\nabla_Y\mathcal{H}\ ,\ \nabla_Y\mathcal{H}\cdot\mathcal{H} = \mathcal{V}\cdot\nabla_Y\mathcal{H};
$$

it helps the intuition to note that for any vector field Y we have:

$$\nabla_Y\mathcal{H} \text{ maps } HM \text{ to } VM \text{ and } VM \text{ to } HM.$$

Differentiation of (3_2) gives a symmetry in X and Y similar to Codazzi's equation:

$$
(3_3)\qquad
\begin{aligned}
\nabla^2_{X,Y}\mathcal{H}\cdot\mathcal{V} - \nabla_Y\mathcal{H}\cdot\nabla_X\mathcal{H} &= \nabla_X\mathcal{H}\cdot\nabla_Y\mathcal{H} + \mathcal{H}\cdot\nabla^2_{X,Y}\mathcal{H}, \quad \text{hence}\\
-\mathcal{H}\cdot\nabla^2_{X,Y}\mathcal{H}\cdot H &= (\nabla_X\mathcal{H}\cdot\nabla_Y\mathcal{H} + \nabla_Y\mathcal{H}\cdot\nabla_X\mathcal{H})\cdot H,\\
+\mathcal{V}\cdot\nabla^2_{X,Y}\mathcal{H}\cdot V &= (\nabla_X\mathcal{H}\cdot\nabla_Y\mathcal{H} + \nabla_Y\mathcal{H}\cdot\nabla_X\mathcal{H})\cdot V.
\end{aligned}
$$

One may retranslate (3) into the less suggestive properties of the O'Neill tensors, this requires many more case distinctions than (3).

Differential equations. First we define along **any** curve $\gamma : I \to M$ vector fields which are called "parallel" in the subbundle HM resp. VM via the first order differential equations

$$
(4_H)\qquad \frac{\nabla^H}{dt}H(t) = 0\ ,\ H(0) \in H_{\gamma(0)}M
$$

$$
(4_V)\qquad \frac{\nabla^V}{dt}V(t) = 0\ ,\ V(0) \in V_{\gamma(0)}M.
$$

Next we want to define "geodesics" for the subbundles by corresponding second order $ODEs$, but $\frac{\nabla^V}{dt}\dot\gamma = 0$ does not make sense before we know for the tangent field that $\dot\gamma(t) \in V_{\gamma(t)}M$. Fortunately the decomposition (2_V) allows us to take instead

$$\frac{\nabla}{dt}\dot\gamma = -\nabla_{\dot\gamma}\mathcal{H}\cdot\dot\gamma\ ,\ \dot\gamma(0) \in V_{\gamma(0)}M. \qquad \text{("geodesic" equation for } VM) \tag{5}$$

Proposition 1.
a) Each solution γ of (5) has $\dot\gamma(t) \in V_{\gamma(t)}M$, i.e. γ stays tangential to the subbundle.
b) If γ is a solution of (5), then so is $\gamma_\lambda(t) := \gamma(\lambda\cdot t)$.

Proof. We will conclude from(5) that $\mathcal{H}\cdot\dot\gamma(t)$ satisfies a linear differential equation; the initial condition is 0 because $\dot\gamma(0) \in V_{\gamma(0)}M$ is assumed in (5); then a), namely $\mathcal{H}\cdot\dot\gamma(t) = 0$, follows. The equation for $\mathcal{H}\cdot\dot\gamma(t)$ is:

$$\frac{\nabla}{dt}\big(\mathcal{H}\cdot\dot\gamma\big) = \nabla_{\dot\gamma}\mathcal{H}\cdot\dot\gamma + \mathcal{H}\cdot\frac{\nabla}{dt}\dot\gamma \overset{(5)}{=} \nabla_{\dot\gamma}\mathcal{H}\cdot\dot\gamma - \mathcal{H}\nabla_{\dot\gamma}\mathcal{H}\cdot\dot\gamma \;=\; \mathcal{V}\nabla_{\dot\gamma}\mathcal{H}\cdot\dot\gamma \overset{(3_2)}{=} \nabla_{\dot\gamma}\mathcal{H}\cdot\big(\mathcal{H}\cdot\dot\gamma\big)$$

For b) note $\dot\gamma_\lambda(t) = \lambda\cdot\dot\gamma(\lambda t)$.
Interpretation. If the subbundle VM were integrable, then (5) would be the geodesic equation for the integral submanifolds. Therefore proposition 1 allows us to call the solutions of (5) "geodesics" of the subbundle VM, whether VM is integrable or not. Note that b) is the reparametrization property of geodesics which is important for the definition of the exponential map, which we generalize to:

$$\exp^V_m(s\cdot\dot\gamma(0)) := \gamma(s), \quad D\exp^V_m|_0 = id|_{V_mM}, \quad \gamma \text{ a solution of (5).}$$

The Frobenius theorem. Let $X_1,\cdots,X_k$ be linearly independent vector fields on a manifold M. They define a subbundle $VM := \mathrm{span}(X_1,\cdots,X_k)$. The Frobenius theorem states: If the integrability condition (6_1) is satisfied then integral submanifolds of VM pass through every point $m \in M$. Sometimes the subbundle VM is given first, e.g. as some eigen distribution; in that case (6_2) is preferred over (6_1).
Frobenius integrability condition:

$$\big[X_i, X_j\big](m) \in V_mM \text{ for all } i,j \text{ and all } m \in M. \tag{6_1}$$

$$\big[U, V\big] \text{ is a vector field in } VM \text{ for all vector fields } U, V \text{ in } VM. \tag{6_2}$$

As soon as one has proved a local version of the theorem one can piece the local integral submanifolds together as in the case of ODEs. One may therefore assume that M is an open subset of R^n, with ∇ standard differentiation. The previous basic setup applies to the situation and the "geodesic equation" (5) suggests the induction free proof of the Frobenius theorem below. If one wishes to consider the Euclidean metric then the pair of complementary projections $\mathcal{H},\mathcal{V}$ is chosen as the natural orthogonal projections. Instead of such a local interpretation one may work in the context of a global pseudo-Riemannian metric g for M; the following proof will then also show how local Euclidean computations can line by line be translated to global Riemannian computations. The main advantage in using a metric (Euclidean or Riemannian) is the possibility of a metric *interpretation* of the orthogonal part of the connection, the actual proof is independent of the metric. We apply the splitting (2_V) to the Lie bracket and get

$$\big[X_i, X_j\big] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = \nabla^V_{X_i} X_j - \nabla^V_{X_j} X_i - \nabla_{X_i}\mathcal{H}\cdot X_j + \nabla_{X_j}\mathcal{H}\cdot X_i.$$

Therefore $(6_{1,2})$ is equivalent to the following tensorial version of the integrability condition:

$$(6_3) \qquad 0 = \mathcal{H}\cdot\big[X_i, X_j\big] = \nabla_{X_j}\mathcal{H}\cdot X_i - \nabla_{X_i}\mathcal{H}\cdot X_j, \quad \text{or : } \ \nabla\mathcal{H} \ \text{ is symmetric on } VM.$$

We thus recognize the Frobenius integrability condition as the symmetry of the second fundamental forms of the expected integral submanifolds. If the connection ∇ is symmetric (as it is in the metric case) then we also note $\nabla^V_U V - \nabla^V_V U - \big[U,\ V\big] = \nabla_U\mathcal{H}\cdot V - \nabla_V\mathcal{H}\cdot U$, so that this skew symmetric part of $\nabla\mathcal{H}$ is the torsion of the connection ∇^V on VM.
We now construct the integral submanifolds. Since proposition 1 identified already the would-be geodesics of these (looked for) submanifolds we define a candidate of an integral manifold through m by mapping a neighbourhood of $0 \in V_mM$ (on which $\exp^V_m$ has maximal rank) with $\exp^V_m$ into M. Then we have to show that all the tangent vectors to this (local) submanifold are indeed in VM. Therefore the Frobenius theorem is proved with the following

Proposition 2.
Let $\gamma_\epsilon(t)$ be a 1-parameter family of solutions of (5) with $\frac{d}{d\epsilon}\gamma_\epsilon(0) \in V_{\gamma_\epsilon(0)}M$ – in our case $\gamma_\epsilon(0) = m$, hence $\frac{d}{d\epsilon}\gamma_\epsilon(0) = 0 \in V_mM$ – then the integrability condition (6) implies

$$\frac{d}{d\epsilon}\gamma_\epsilon(t) \in V_{\gamma_\epsilon(t)}M.$$

In other words: Curves $\epsilon \to \gamma_\epsilon(t)$ in families of solutions $t \to \gamma_\epsilon(t)$ of (5) stay tangential to the subbundle VM.

Proof. We derive a linear differential equation for $t \to \mathcal{H}\cdot\frac{d}{d\epsilon}\gamma_\epsilon(t)$; the initial condition is zero because of the assumption $\frac{d}{d\epsilon}\gamma_\epsilon(0) \in V_{\gamma_\epsilon(0)}M$; this gives $\mathcal{H}\cdot\frac{d}{d\epsilon}\gamma_\epsilon(t) \equiv 0$.
Abbreviate $\frac{d}{dt}\gamma_\epsilon(t) = \dot\gamma_\epsilon(t)$.

$$\begin{aligned}\frac{\nabla}{dt}\left(\mathcal{H}\cdot\frac{d}{d\epsilon}\gamma_\epsilon(t)\right) &= \nabla_{\dot\gamma_\epsilon}\mathcal{H}\cdot\frac{d}{d\epsilon}\gamma_\epsilon(t) + \mathcal{H}\cdot\frac{\nabla}{dt}\frac{d}{d\epsilon}\gamma_\epsilon(t)\\ &= \nabla_{\dot\gamma_\epsilon}\mathcal{H}\cdot(\mathcal{H}+\mathcal{V})\frac{d}{d\epsilon}\gamma_\epsilon(t) + \mathcal{H}\cdot\frac{\nabla}{d\epsilon}\frac{d}{dt}\gamma_\epsilon(t).\end{aligned}$$

In the second term we can use $\frac{d}{dt}\gamma_\epsilon(t) \in V_{\gamma_\epsilon(t)}M$, hence with (2_V)

$$\mathcal{H}\cdot\frac{\nabla}{d\epsilon}\frac{d}{dt}\gamma_\epsilon(t) = -\nabla_{\frac{d}{d\epsilon}\gamma_\epsilon}\mathcal{H}\cdot\dot\gamma_\epsilon = -\nabla_{(\mathcal{H}+\mathcal{V})\frac{d}{d\epsilon}\gamma_\epsilon}\mathcal{H}\cdot\dot\gamma_\epsilon.$$

We sort terms and use the symmetry (integrability) condition (6_3), i.e. $\nabla_{\mathcal{V}\frac{d}{d\epsilon}\gamma_\epsilon}\mathcal{H}\cdot\dot\gamma_\epsilon = \nabla_{\dot\gamma_\epsilon}\mathcal{H}\cdot\mathcal{V}\frac{d}{d\epsilon}\gamma_\epsilon$, to get the desired differential equation involving $\nabla\mathcal{H}$ and $\dot\gamma_\epsilon$,

$$\frac{\nabla}{dt}\left(\mathcal{H}\cdot\frac{d}{d\epsilon}\gamma_\epsilon(t)\right) = \nabla_{\dot\gamma_\epsilon}\mathcal{H}\cdot\left(\mathcal{H}\frac{d}{d\epsilon}\gamma_\epsilon(t)\right) - \nabla_{\mathcal{H}\cdot\frac{d}{d\epsilon}\gamma_\epsilon(t)}\mathcal{H}\cdot\dot\gamma_\epsilon(t).$$

Remark. This computation gives an interpretation of the skewsymmetric part of the second fundamental form of a non-integrable subbundle of TM:

$$\frac{\nabla}{dt}\left(\mathcal{H}\cdot\frac{d}{d\epsilon}\gamma_\epsilon(t)\right)|_{t=0} = \nabla_{\dot\gamma_\epsilon(0)}\mathcal{H}\cdot\left(\mathcal{V}\frac{d}{d\epsilon}\gamma_\epsilon(0)\right) - \nabla_{\mathcal{V}\cdot\frac{d}{d\epsilon}\gamma_\epsilon(0)}\mathcal{H}\cdot\dot\gamma_\epsilon(0).$$

The skewsymmetric part thus determines the speed with which $\frac{d}{d\epsilon}\gamma_\epsilon(t)|_{t=0}$ starts to rotate out of the bundle VM.

Submersions. Start with a fibre bundle $\pi : M \to B$ with a pseudo-Riemannian metric g on the total space M of the bundle. As the fibres of the subbundle VM of our basic setup one takes the tangent spaces of the fibres $\pi^{-1}(b)$ of π. All previous considerations apply, in particular (6_3) holds. For submersions one has additional data, namely a pseudo-Riemannian metric g^B with connection ∇^B on the base B, and the following submersion assumption:

(7) $$D\pi|_{H_mM} : H_mM \longrightarrow T_{\pi(m)}B \text{ are isometries for all } m\in M.$$

The fibre bundle structure invites the use of the so called basic vector fields, namely, given a vector field ξ on B there is exactly one vector field H in HM such that $D\pi_m(H) = \xi(\pi(m))$ for all $m \in M$. We write $H = \hat\xi$. These are a usefull tool even without (7), in the submersion case they help to prove – in addition to (6_3) – another symmetry property of $\nabla\mathcal{H}$:

Proposition 3.

a) For basic $\hat\xi$ in HM and (fibretangential) V in VM holds (without metric assumptions)

$$D\pi([\hat\xi, V]) = 0 \ \text{ or } \ \mathcal{H}\cdot[\hat\xi, V] = 0.$$

b) The vertical change of the metric g on basic fields is given by
$$d_V(g(\hat\xi,\hat\eta)) = -g(\nabla_{\hat\xi}\mathcal{H}\cdot\hat\eta + \nabla_{\hat\eta}\mathcal{H}\cdot\hat\xi, V).$$
Assuming (7) this is 0, hence
$\nabla\mathcal{H}$ is skewsymmetric on HM, $\nabla_H\mathcal{H}\cdot K + \nabla_K\mathcal{H}\cdot H = 0$.

c) Assuming (7), then geodesics in M which start orthogonal to fibres $\pi^{-1}(b)$, remain orthogonal to fibres of π (i.e. remain tangential to HM, compare proposition 1).

Remark. The following proof of proposition 3 is included for convencience; it differs from the standard version mainly in its emphasis on the tensorfield $\nabla\mathcal{H}$. Also, recall the definition of the hessian of a map $F : \ M \to \bar M$: Let $t \to m(t)$ be a curve in M and

$t \to X(t) \in T_{m(t)}M$ a vector field along the curve, then $t \to DF_{m(t)} \cdot X(t) \in T_{F(m(t))}\bar{M}$ is indeed a vector field along the image curve and

$$\nabla^2 F\big(\frac{d}{dt}m(t), X(t)\big) = \frac{\bar{\nabla}}{dt}\big(DF_{m(t)} \cdot X(t)\big) - DF \cdot \frac{\nabla}{dt}X(t).$$

Appealing to taste again: If one prefers one may use the shorter formula

$$\nabla^2 F(X,Y) \ := \bar{\nabla}_{DF\cdot X}\big(DF \cdot Y\big) - DF \cdot \nabla_X Y,$$

in which the first term on the right needs additional explanations.

Proof. a) The statement does not involve the connection, so we compute with a symmetric derivative ∇ (i.e. $\big[X,Y\big] = \nabla_X Y - \nabla_Y X$ and the second derivative of maps, $\nabla^2 F$, is symmetric). Let $m(t)$ be an integral curve of V, then we have $D\pi \cdot \hat{\xi}(m(t)) = \text{const} \in T_{\pi(m)}B$, hence

$$(8_1) \qquad \nabla^2\pi(\dot{m}, \hat{\xi}) + D\pi(\nabla_{\dot{m}}\hat{\xi}) = 0.$$

On the other hand, let $h(t)$ be any curve in M, in particular an integral curve of $\hat{\xi}$, then $D\pi \cdot V(h(t)) = 0 \in T_{\pi(h(t))}B$, hence

$$(8_2) \qquad \nabla^2\pi(\dot{h}, V) + D\pi(\nabla_{\dot{h}}V) = 0.$$

The symmetry of the second covariant derivative of maps, here of $\nabla^2\pi$, together with (8) gives

$$D\pi(\nabla_V\hat{\xi} - \nabla_{\hat{\xi}}V) = 0,$$

hence a). We use this now in the form $\mathcal{H} \cdot \nabla_V\hat{\xi} = \mathcal{H} \cdot \nabla_{\hat{\xi}}V = -\nabla_{\hat{\xi}}\mathcal{H} \cdot V$ to compute the derivative of scalar products of basic fields:

$$\begin{aligned}
d_V(g(\hat{\xi},\hat{\eta})) &= g(\nabla_V\hat{\xi},\hat{\eta}) + g(\hat{\xi},\nabla_V\hat{\eta}) \\
\text{(Proposition 3}a) &= g(\nabla_{\hat{\xi}}V,\hat{\eta}) + g(\hat{\xi},\nabla_{\hat{\eta}}\hat{V}) \\
(2_V) &= -g(\nabla_{\hat{\xi}}\mathcal{H}\cdot V,\hat{\eta}) - g(\hat{\xi},\nabla_{\hat{\eta}}\mathcal{H}\cdot V) \\
\text{(symmetry of } \nabla_{\hat{\xi}}\mathcal{H},\ 3_1) &= -g(\nabla_{\hat{\xi}}\mathcal{H}\cdot\hat{\eta} + \nabla_{\hat{\eta}}\mathcal{H}\cdot\hat{\xi}, V),
\end{aligned}$$

hence b). To prove c) we only have to apply proposition 1 to the subbundle HM instead of VM. The skewsymmetry b) then says that the solutions of (5) are geodesics in M.

Remarks. (i) The relations with the second derivatives of π are derived as follows:
If one argument is vertical then differentiate $D\pi(V) = 0$ as in (8_2). For horizontal arguments we have to use (7): Let γ be the horizontal lift of a shortest geodesic $b = \pi \circ \gamma$ in B, both have the same length; if γ were not a shortest geodesic then the projection of a shorter replacement would shorten $\pi \circ \gamma$ in B, reproving proposition 3c. Hence

$$(9) \qquad \begin{aligned}
&\nabla^2\pi(\dot{\gamma},\dot{\gamma}) = 0\ ,\ \nabla^2\pi(H,K) = 0 \qquad\qquad \text{(by bilinearization)} \\
&\nabla^2\pi(X,V) = -D\pi(\nabla_X V) \overset{(2_V)}{=} D\pi(\nabla_X\mathcal{H}\cdot V)
\end{aligned}$$

(ii) While symmetries of tensor fields carry over to their derivatives as in (3), additional terms complicate the derivative of partial symmetries. Differentiate $\nabla_U\mathcal{H}\cdot V = \nabla_V\mathcal{H}\cdot U$ using $\nabla^V_X U = 0 = \nabla^V_X V$ to get

$$\begin{aligned}&\nabla^2_{X,U}\mathcal{H}\cdot V - \nabla_{\nabla_X\mathcal{H}\cdot U}\mathcal{H}\cdot V - \nabla_U\mathcal{H}\cdot\nabla_X\mathcal{H}\cdot V =\\ &\nabla^2_{X,V}\mathcal{H}\cdot U - \nabla_{\nabla_X\mathcal{H}\cdot V}\mathcal{H}\cdot U - \nabla_V\mathcal{H}\cdot\nabla_X\mathcal{H}\cdot U.\end{aligned}$$

Curvature computations. The final formulas are (10), (11) and (assuming (7)), (14).
MIXED CURVATURES. Most of the "mixed" curvature identities (i.e. with vectors from HM as well as VM as arguments of the curvature tensor R of ∇) come from the product rule:

$$\begin{aligned}R(X,Y)(\mathcal{H}\cdot Z) &= (\nabla^2_{X,Y} - \nabla^2_{Y,X})(\mathcal{H}\cdot Z) && \text{(definition of } R\text{)}\\ &= (\nabla^2_{X,Y} - \nabla^2_{Y,X})\mathcal{H}\cdot Z + \mathcal{H}\cdot R(X,Y)Z && \text{(product rule)},\end{aligned}$$

or, with more special arguments:

$$\begin{aligned}\mathcal{V}\cdot R(X,Y)H &= (\nabla^2_{X,Y}\mathcal{H} - \nabla^2_{Y,X}\mathcal{H})\cdot H, \text{ or}\\ -\mathcal{H}\cdot R(X,Y)V &= (\nabla^2_{X,Y}\mathcal{H} - \nabla^2_{Y,X}\mathcal{H})\cdot V.\end{aligned}\tag{10}$$

Some of the complementary components of R are obtained with the Bianchi identity since the right sides of (10_B) are already expressed in (10):

$$\begin{aligned}\mathcal{H}\cdot R(U,V)H &= \mathcal{H}\cdot R(U,H)V + \mathcal{H}\cdot R(H,V)U,\\ \mathcal{V}\cdot R(H,K)V &= \mathcal{V}\cdot R(H,V)K + \mathcal{V}\cdot R(V,K)H,\\ \mathcal{H}\cdot\big(R(U,H)K - R(U,K)H\big) &= \mathcal{H}\cdot R(K,H)U,\\ \mathcal{V}\cdot\big(R(H,U)V - R(H,V)U\big) &= \mathcal{V}\cdot R(V,U)H.\end{aligned}\tag{10_B}$$

(See (3_3) for the vertical part $\mathcal{V}\cdot\nabla^2_{X,Y}\mathcal{H}\cdot V$ and the horizontal part $\mathcal{H}\cdot\nabla^2_{X,Y}\mathcal{H}\cdot H$.)
THE REMAINING MIXED CURVATURES. Differentiate (2_V) : $\nabla_Y V = \nabla^V_Y V - (\nabla_Y\mathcal{H})\cdot V$ once more to obtain (by the usual computation with difference tensors of connections):

$$R(X,Y)V = R^V(X,Y)V - (\nabla^2_{X,Y}\mathcal{H} - \nabla^2_{Y,X}\mathcal{H})\cdot V - (\nabla_X\mathcal{H}\cdot\nabla_Y\mathcal{H} - \nabla_Y\mathcal{H}\cdot\nabla_X\mathcal{H})\cdot V,$$

and after adding (10) this gives the second set of curvature formulas:

$$\mathcal{V}\cdot R(X,Y)V = R^V(X,Y)V - (\nabla_X\mathcal{H}\cdot\nabla_Y\mathcal{H} - \nabla_Y\mathcal{H}\cdot\nabla_X\mathcal{H})\cdot V.\tag{11}$$

If $X, Y \in VM$ then (11) are the Gauß equations of the fibres of π.
The curvature tensor R^H of the connection ∇^H of the (usually) non-integrable bundle HM is less popular, but of course one has – analogous to (11) –

$$\mathcal{H}\cdot R(X,Y)H = R^H(X,Y)H - \big[\nabla_X\mathcal{H}, \nabla_Y\mathcal{H}\big]\cdot H.\tag{12}$$

We have expressed $R(H,K)V$ and $R(U,V)H$ in terms of $\nabla^2\mathcal{H}$, see $(10, 10_B)$, hence we get the corresponding values of R^V, R^H from (11,12), for example:

$$R^V(H,K)V = (\nabla^2_{V,K}\mathcal{H} - \nabla^2_{K,V}\mathcal{H})H + (\nabla^2_{H,V}\mathcal{H} - \nabla^2_{V,H}\mathcal{H})K + \big[\nabla_H\mathcal{H}, \nabla_K\mathcal{H}\big]V.$$

Similarly one can express the skewsymmetric part $R^V(H,U)V - R^V(H,V)U$ in terms of $\nabla\mathcal{H}, \nabla\mathcal{H}^2$.
Berard Bergery has an example where $\nabla\mathcal{H} = 0$, so that all the curvatures mentioned so far are zero, but $\mathcal{H}\cdot R(U,H)H$ and $\mathcal{V}\cdot R(H,V)V$ are not. In that case the projections are not symmetric for the metric g, so that $\mathcal{H}\cdot R(U,H)H$ cannot be recovered from $g(R(U,H)H,K)$, but $g(R(U,H)H,U)$ is not computable from $\nabla\mathcal{H}, \nabla^2\mathcal{H}$. (For symmetric projections, (10) would give the answer.)
Curvatures on horizontal vectors. As a last step it remains to relate R on horizontal vectors to the curvature tensor R^B of the base. In general this will involve the metric distortion of $D\pi$ on HM; we use now the *submersion assumption* (7) heavily. Let η, κ, λ be vector fields on B and $H = \hat\eta$, $K = \hat\kappa$, $L = \hat\lambda$ their horizontal lifts. We differentiate $D\pi(L) = \lambda$ and use (7) via (9), namely $\nabla^2\pi(\mathcal{H}\cdot, \mathcal{H}\cdot) = 0$, to get

$$(13_1) \qquad D\pi(\nabla_K L) = \nabla^B_\kappa\lambda\,, \quad \text{hence } D\pi(\big[K,L\big]) = \big[\kappa,\lambda\big].$$

Differentiate again and use $\nabla^2\pi(\mathcal{H}\cdot, \mathcal{H}\cdot) = 0$:

$$\nabla^2\pi(H, \mathcal{V}\cdot\nabla_K L) + D\pi(\nabla_H\nabla_K L) = \nabla^B_\eta\nabla^B_\kappa\lambda.$$

Insert $\mathcal{V}\cdot\nabla_K L = \nabla_K\mathcal{H}\cdot L$, see (2_H), and $\nabla^2\pi(H,V) = D\pi(\nabla_H\mathcal{H}\cdot V)$, see (9), to get:

$$(13_2) \qquad D\pi(\nabla_H\mathcal{H}\cdot\nabla_K\mathcal{H}\cdot L) + D\pi(\nabla_H\nabla_K L) = \nabla^B_\eta\nabla^B_\kappa\lambda.$$

Next, (13_1) gives $D\pi(\nabla_{\mathcal{H}\cdot[H,K]}L) = \nabla^B_{[\eta,\kappa]}\lambda$, hence

$$D\pi(\big[\nabla_H\mathcal{H}, \nabla_K\mathcal{H}\big]\cdot L) + D\pi(\nabla_{\mathcal{V}\cdot[H,K]}L) + D\pi(R(H,K)L) = R^B(\eta,\kappa)\lambda.$$

In the second term use proposition 3 to eliminate $\mathcal{V}\cdot[H,K]$:

$$D\pi(\nabla_{\mathcal{V}\cdot[H,K]}L) \overset{3a)}{=} D\pi(\nabla_L(\mathcal{V}\big[H,K\big])) \overset{(2_V)}{=} -D\pi(\nabla_L\mathcal{H}\cdot\mathcal{V}\big[H,K\big]) \overset{3b)}{=} -2\cdot D\pi(\nabla_L\mathcal{H}\cdot\nabla_H\mathcal{H}\cdot K),$$

which gives finally the often used result:

$$(14) \qquad \begin{aligned} R^B(\eta,\kappa)\lambda &= D\pi(R(H,K)L + \big[\nabla_H\mathcal{H}, \nabla_K\mathcal{H}\big]\cdot L - 2\nabla_L\mathcal{H}\cdot\nabla_H\mathcal{H}\cdot K) \\ g^B(R^B(\eta,\kappa)\kappa,\eta) &= g(R(H,K)K,H) + 3\cdot g(\nabla_H\mathcal{H}\cdot K, \nabla_H\mathcal{H}\cdot K). \end{aligned}$$

The standard deformation. As metric space the base B is the set of fibres of π, often visualized in M by shrinking the size of the fibres of π to zero. Without using the above submersion description we approach this limit with the following "canonical" deformation of the metric of M (15). The limit curvature is the curvature of the base. This is an approach to homogenous spaces assuming they are constructed as sets of orbits with the induced distance in HM between the orbits (as disjoint sets) *before* the fibre bundle structure of that situation is considered. Note that the difference tensors Γ^ϵ between the occuring Levi-Civita connections ∇^ϵ are surprisingly simple, see (16), if the projections $\mathcal{H}, \mathcal{V}$ are symmetric.

$$
\begin{aligned}
g_\epsilon(X,Y) &:= g((id-\epsilon\mathcal{V})\cdot X, (id-\epsilon\mathcal{V})\cdot Y), && 0 \le \epsilon < 1,\\
\nabla^\epsilon_Y Z &= \nabla_Y Z + \Gamma^\epsilon(Y,Z), && \text{(definition of } \Gamma^\epsilon\text{)},\\
(\nabla_X g_\epsilon)(Y,Z) &= g_\epsilon(\Gamma^\epsilon(X,Y),Z) + g_\epsilon(Y,\Gamma^\epsilon(X,Z))\\
&= -\epsilon\cdot g(\nabla_X\mathcal{V}\cdot Y, (id-\epsilon\mathcal{V})Z) - \epsilon\cdot g((id-\epsilon\mathcal{V})\cdot Y, \nabla_X\mathcal{V}\cdot Z).
\end{aligned}
\tag{15}
$$

This leads to simple expressions for Γ^ϵ which shed additional light on (14).
If $X,Y,Z \in VM$ or $X,Y,Z \in HM$ then $(\nabla_X g_\epsilon)(Y,Z) = 0$, hence

$$\mathcal{V}\cdot\Gamma^\epsilon(\mathcal{V}\cdot,\mathcal{V}\cdot) = 0\ ,\ \mathcal{H}\cdot\Gamma^\epsilon(\mathcal{H}\cdot,\mathcal{H}\cdot) = 0. \tag{16_1}$$

If one puts $Y,Z \in HM$, $X \in VM$ into (15) then one finds with proposition 3b), namely $\nabla_Y\mathcal{V}\cdot Z = -\nabla_Z\mathcal{V}\cdot Y$, that the difference tensors vanish on horizontal pairs:

$$\Gamma^\epsilon(Y,Z) = 0 \quad \text{for} \quad Y,Z \in HM. \tag{16_2}$$

Finally use $Y \in HM$, $Z \in VM$, $X = \mathcal{H}\cdot X + \mathcal{V}\cdot X$ in (15); we get after a short computation, which uses both symmetries, namely (6) (VM is integrable) and proposition 3b) (each g_ϵ is a submersion metric):

$$\Gamma^\epsilon(Z,Y) = \Gamma^\epsilon(Y,Z) = -\epsilon\cdot(2-\epsilon)\cdot\nabla_Y\mathcal{V}\cdot Z = \epsilon\cdot(2-\epsilon)\cdot\nabla_Y\mathcal{H}\cdot Z \in HM. \tag{16_3}$$

With this difference tensor the standard curvature computation (we start, because of (16_2), from $\nabla^\epsilon_H(\nabla^\epsilon_K L) = \nabla_H(\nabla_K L) + \Gamma^\epsilon(H, \mathcal{V}\cdot\nabla^\epsilon_K L)$) gives

$$R^\epsilon(H,K)L = R(H,K)L + \epsilon(2-\epsilon)\cdot\big(\big[\nabla_H\mathcal{H},\nabla_K\mathcal{H}\big]\cdot L - 2\nabla_L\mathcal{H}\cdot\nabla_H\mathcal{H}\cdot K\big). \tag{17}$$

A comparison of (14) and (17) gives the suggestive result: The curvature $R^1(H,K)L$ of the limiting symmetric connection ∇^1 (for the degenerate metric g_1 on M) is isometrically mapped by $D\pi$ to the curvature $R^B(D\pi\cdot H, D\pi\cdot K)D\pi\cdot L$ of the base B. – Note that the above standard deformation may be continued to pseudo-Riemannian metrics with a different signature but the same connections since $\Gamma^\epsilon = \Gamma^{2-\epsilon}$.

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