MAXIMAL RESTRICTION ESTIMATES AND THE MAXIMAL FUNCTION OF THE FOURIER TRANSFORM

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Abstract. We prove a maximal restriction inequality for the Fourier transform, providing an answer to a question left open by Müller, Ricci and Wright [7]. Our methods are similar to the ones in [7] and [1], with the addition of a suitable trick to help us linearise our maximal function. In the end, we comment on how to use the same trick in combination with Vitturi's approach [11] to obtain a stronger high-dimensional result.

1. Introduction

Restriction estimates for the Fourier transform have been a very active topic within harmonic analysis for over the past 40 years. Basically, one inquires whether an inequality of the form

$$\|\widehat{f}|_S\|_{L^q(S,\mathrm{d}\sigma)} \le C_{p,d}\|f\|_{L^p(\mathbb{R}^d)} \tag{1}$$

can hold on a hypersurface S, where σ stands for the standard surface measure on S, which is the same as the arclength measure for the case of plane curves. Here we shall focus on compact hypersurfaces S with non-vanishing curvature, the typical example being the sphere $\mathbb{S}^{d-1}$. By taking examples of functions (either the so called *Knapp* examples or constant functions; see, e.g., [9, Section 4]), one finds out that a *necessary* condition for such inequalities to hold is that

$$1 \le p < \frac{2d}{d+1} \text{ and } p' \ge \frac{d+1}{d-1}q, \tag{2}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. The *restriction conjecture* then asserts that the above conditions are also *sufficient.*

The first manifestation of such a restriction principle, in a range smaller than (2), was perhaps the result of Fefferman and Stein (see [2, page 28]), where an estimate in all dimensions for $q=2$ was proven, this estimate being sharpened to the optimal range of p for such q by Tomas [10], who credits Stein for the endpoint result. For the sphere (and, in general, for compact hypersurfaces with non-vanishing curvature), it reads that

$$\|\widehat{f}|_S\|_{L^2(S,d\sigma)} \le C_{p,d}\|f\|_{L^p(\mathbb{R}^d)},$$

whenever $1 \le p \le \frac{2(d+1)}{d+3}$.

Regarding ranges of exponents, for dimension $d \ge 3$, Problem (1) is still open, with new technology being developped continously to improve ranges of exponents;

see, for instance, [9, 4, 5] for further developments in this subject.

For dimension 2, however, Problem (1) has been completely solved, as we observe that the conditions can be rewritten as follows:

$$1 \le p < \frac{4}{3},\ p' \ge 3q. \tag{3}$$

In the non-endpoint case $p' > 3q$, the result is due to Fefferman [2, page 33], and the endpoint to Zygmund [12] and Carleson and Sjölin [1]. Later, Sjölin [8] also extended these results to other classes of curves.

In [7], D. Müller, F. Ricci and J. Wright consider a slight strenghtening of the restriction properties of the Fourier transform in two dimensions: namely, they prove a maximal version of restriction estimates and conclude a differentiation result. Here, we shall state the result only in the case of $\mathbb{S}^1$, for simplicity:

Theorem 1. *[Müller, Ricci, Wright [7]; 2016] Let $\mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{C}$ be a L^p function. Assume that $1 \le p < \frac{8}{7}$. Then, with respect to arclength measure, almost every point in $\mathbb{S}^1$ is a Lebesgue point for $\widehat{f}$ and the regularised value of $\widehat{f}$ at x coincides with the restriction operator $\mathcal{R}f(x)$ for almost every $x \in \mathbb{S}^1$.*

The purpose of this note is to improve ranges of exponents of such maximal restriction results. Explicitly, our main result is:

Theorem 2. *Theorem 1 extends to $1 \le p < \frac{4}{3}$.*

The strategy in [7] passes through a maximal function with absolute values *outside* the integral, and then uses Hölder inequality. Namely, they focus on maximal functions of the form

$$\mathcal{M}f(x) = \sup_{\substack{R \text{ axis parallel,} \\ \text{centered at} x}} \left| \int \chi_R(y) \widehat{f}(y)\, \mathrm{d}y \right|,$$

where $\chi_R \in \mathcal{S}(\mathbb{R})$ is a smooth bump function adapted to the rectangle R. They then prove that, for the *whole* restriction range $1 \le p < \frac{4}{3}$ and $p' \ge 3q$,

$$\|\mathcal{M}f\|_{L^q(\mathrm{d}\sigma)} \le C_{p,\Gamma} \|f\|_{L^p(\mathbb{R}^2)},$$

where σ stands again for the arclength measure on the curve Γ. Finally, in order to prove Theorem 1, the authors bound the maximal function

$$M_{\mathcal{R}}f(t) = \sup_{\substack{R \text{ axis parallel,} \\ \text{centered at} x}} \int \chi_R(y) |\widehat{f}(y)|\, \mathrm{d}y \tag{4}$$

by $(\mathcal{M}h(x))^{1/2}$, where $h = f * \tilde{f}$, with $\tilde{f}(x,y) = f(-x,-y)$.

In order to prove Theorem 2, it suffices to bound (4) from $L^r(\mathbb{R}^2)$ to some $L^q(\mathbb{S}^1)$, where $1 \le r < \frac{4}{3}$, as the stated property holds trivially in the class $\mathcal{S}(\mathbb{R}^2)$. By nature of such an approximation argument, it sufficies to prove these bound for functions $f \in \mathcal{S}(\mathbb{R}^2)$.

For fixed g with $\|g\|_\infty = 1$ and measurable choice R of axis-parallel rectangles, define the linearised maximal operator

$$M_{g,R}f(x) = \int_{\mathbb{R}^2} |R(x)|^{-1} 1_{R(x)}(y-x)\widehat{f}(y)g(y)\,\mathrm{d}y \tag{5}$$

acting initially, say, on functions in $L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$. Setting $g(y) = \frac{\overline{\widehat{f}}}{|\widehat{f}|}$ where $\widehat{f} \neq 0$, and zero otherwise, together with measurable R such that $M_{g,R}f(t) \geq \frac{1}{2}M_{\mathcal{R}}f$, implies that it is in turn sufficient to estimate (5) from $L^r(\mathbb{R}^2)$ to some $L^q(\mathbb{S}^1)$. This is the basic goal of Lemmata 1 and 2.

Following [7], M. Vitturi [11] and V. Kovac and D. Oliveira e Silva [6] have proved, as a consequence of $p' = 4$ being admissible for the restriction estimate, results in dimensions ≥ 3 : they have obtained that, in the same range of exponents as in Theorem 4, one gets *pointwise convergence* $\chi_\varepsilon * \widehat{f} \to \widehat{f}$ for $\sigma-$almost every point on the sphere $\mathbb{S}^{d-1}$, where $\chi_\varepsilon(y) = \frac{1}{\varepsilon^n}\chi(y/\varepsilon)$, and $\chi \in \mathcal{S}(\mathbb{R}^d)$. Although this is already present in [11] and in both cases the techniques also imply the same result for $\chi = 1_{B(0,1)}$, the ideas in [6] represent a stronger, quantitative form of such a theorem, as they consider *variation norms* instead of suprema.

Our second result is also an improvement on Vitturi's techniques, yet in another direction:

Theorem 3. *Let $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \frac{4}{3}$. Then $\sigma-$almost every point of $\mathbb{S}^{d-1}$ is a Lebesgue point of $\widehat{f}$, and the regularised value of $\widehat{f}$ at x coincides with the restriction operator $\mathcal{R}f(x)$ for almost every $x \in \mathbb{S}^{d-1}$.*

The argument to prove Theorem 3 is similar to the one employed to treat Theorem 2, and we postpone it to the end of this manuscript.

2. Main Argument

Call a measurable function a in $\mathbb{R}^d$ *bump function* if there exists an axis parallel rectangle R centered at the origin with

$$|a| \leq |R|^{-1}1_R.$$

Convolution with such a bump function satisfies a pointwise bound by the strong Hardy Littlewood maximal function, uniformly in the rectangle. The following lemma concerns an adjoint of a linearised maximal operator, combined with a Fourier transform.

Lemma 1. *For each $x \in \mathbb{R}^d$ let a_x be a convolution product of k bump functions. Assume further that $a_x(y)$, as function in (x, y), is in $L^\infty(x, L^1(y))$. Let T be defined on functions $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ by*

$$Tf(\xi) = \int_{\mathbb{R}^d} \widehat{a}_x(\xi)e^{2\pi i x\cdot\xi}f(x)\,\mathrm{d}x.$$

Then, for some universal constant C depending on k and d only,

$$\|Tf\|_2 \leq C\|f\|_2.$$

Proof. We set up a duality argument, testing Tf against an arbitrary function $\widehat{g} \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. We have, by Fubini and Plancherel,

$$\int_{\mathbb{R}^d} \overline{\widehat{g}(\xi)} \int_{\mathbb{R}^d} \widehat{a}_x(\xi) e^{2\pi i x\cdot\xi} f(x)\,\mathrm{d}x\mathrm{d}\xi = \int_{\mathbb{R}^d} f(x) \int_{\mathbb{R}^d} \overline{g(y)} a_x(y-x)\,\mathrm{d}y\mathrm{d}x$$

Identifying on the right-hand-side a k fold convolution by bump functions acting on g, we estimate the last display by

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x)| M^k(g)(x)\,\mathrm{d}x \le \|f\|_2 \|M^k g\|_2 \le C\|f\|_2 \|\widehat{g}\|_2\ ,$$

where we have used the strong maximal theorem and Plancherel again. Since $\widehat{g}$ was arbitrary, this proves Lemma 1.

□

The hypotheses in the next Lemma are motivated by the parameterised circle

$$z(t) = (\cos(2\pi t), \sin(2\pi t)).$$

By the addition theorem for the sine function, we have

$$|\det(z'(t), z'(s))| = 4\pi^2 |\sin(2\pi(s-t))|.$$

Note the vanishing of the determinant when the two tangent vectors become parallel or anti-parallel. Note further that one can recover $t \neq s \in I := [0, 1)$ from

$$x := z(t) + z(s).$$

Namely, $x/2$ is the midpoint between $z(t)$ and $z(s)$, and these two points on the circle are mirror symmetric relative to the line through this midpoint and the origin. This determines the two points $t \neq s$, up to permutation. Define, therefore, the *upper triangle*

$$\Delta = \{(t, s) \in I \times I : t > s\}.$$

Lemma 2. *Let $z : \mathbb{R} \to \mathbb{R}^2$ be a smooth one-periodic curve such that for all $(t, s) \in \Delta$*

$$|\det(z'(t), z'(s))| \ge c|\sin(2\pi(t-s))| \tag{6}$$

and such that the map

$$(t, s) \to z(t) + z(s) \tag{7}$$

is a bijection from Δ onto a bounded set $\Omega \subset \mathbb{R}^2$. With $a_{z(t)}$ a bump function for every $t \in I$ such that $a_{z(t)}(x)$ is in $L^\infty(t, L^1(x))$, consider an operator acting on functions in $L^4(I)$ as follows:

$$Tf(\xi) = \int_I \widehat{a}_{z(t)}(\xi) e^{2\pi\xi\cdot z(t)} f(t)\,\mathrm{d}t.$$

Then we have for all $1 \le p < 2$ with some constant depending only on p:

$$\|Tf\|_{2p'} \le C_p \|f\|_{\frac{2p}{3-p}},$$

with the obvious interpretation when $p' = \infty$. Notice, moreover, that the reciprocals $\left(\frac{1}{2p'}, \frac{3-p}{2p}\right)$ of the aforementioned exponents lie on the line segment joining $(1/4, 1/4)$ and $(0, 1)$.

Proof. To reduce to Lemma 1, we need to pass to a two dimensional integral. We follow the idea of Carleson-Sjölin and consider the square

$$Tf(\xi)^2 = \int_{I\times I} \widehat{a}_{z(t)}(\xi)\widehat{a}_{z(s)}(\xi)e^{2\pi\xi\cdot(z(t)+z(s))}f(t)f(s)\,\mathrm{d}t\mathrm{d}s.$$

The integral is twice the analoguous integral over the triangle Δ, where we change coordinates by the bijective map (7) to obtain

$$Tf(\xi)^2 = 2\int_\Omega \widehat{b}_x(\xi)e^{2\pi i\xi\cdot x}g(x)\,\mathrm{d}x.$$

Here we have unambiguously defined, for (t,s) in the triangle,

$$\widehat{b}_{z(t)+z(s)} := \widehat{a}_{z(t)}\widehat{a}_{z(s)},$$

$$g(z(t)+z(s)) := f(t)f(s)|\det(z'(t),z'(s))|^{-1}.$$

Note that the determinant here is the Jacobian determinant of the map (7).

It is now easy to prove, by interpolation, that for $1\le p\le 2$ we have

$$\|Tf\|_{2p'}^{2p} = \|(Tf)^2\|_{p'}^{p} \le C\|g\|_p^p\ .$$

Namely, $p=2$ follows directly from Lemma 1 applied to a function supported on Ω, and $p=1$ is trivial since $\|\widehat{b}_x\|_\infty \le C$. To conclude the proof of the lemma, we invert the change of variables to estimate the right-hand-side for $1\le p<2$:

$$\int_\Omega |g(x)|^p\,\mathrm{d}x = \int_\Delta |f(t)f(s)|^p|\det(z'(t),z'(s))|^{1-p}\,\mathrm{d}t\mathrm{d}s \le C_p\||f|^p\|_{\frac{2}{3-p}}^2 = C_p\|f\|_{\frac{2p}{3-p}}^{2p}.$$

Here, the last inequality follows from the Hardy–Littlewood–Sobolev inequality for fractional integrals. Namely, we estimate with (6) on the triangle:

$$|\det(z'(t),z'(s))|^{1-p} \le C\sum_{k=-2}^{2}|t-s-k|^{1-p},$$

and we note that each summand leads to a translated fractional integral. □

Proof of Theorem 2. We introduce the bump function

$$a_x(y) := |R(x)|^{-1}1_{R(x)}(y)\overline{g(x-y)},$$

and write

$$M_{g,R}f(t) = \int_{\mathbb{R}^2}\overline{a_{z(t)}}(y-z(t))\widehat{f}(y)\,\mathrm{d}y.$$

This is just a composition of the operator in (5) with a parametrisation, so we identify them. By Plancherel, similarly to the proof of Lemma 1, we have

$$M_{g,R}f(t) = \int_{\mathbb{R}^2}\overline{\widehat{a}_{z(t)}}(\xi)e^{-2\pi i\xi\cdot z(t)}f(\xi)\,\mathrm{d}y.$$

The adjoint operator then becomes

$$M_{g,R}^*(h)(\xi) = \int_I \widehat{a}_{z(t)}(\xi)e^{2\pi i\xi\cdot z(t)}h(t)\,\mathrm{d}t.$$

By Lemma 2, this is bounded from $L^{\frac{2p}{3-p}}$ to $L^{2p'}$ for $p<2$. We set now $r=(2p')'$. By a computation, $\frac{2p}{3-p}=(r'/3)'$. With this notation, we have that $M_{g,R}$ is bounded from $L^r(\mathbb{R}^2)$ to $L^{r'/3}(\mathbb{S}^1)$ for all $r < \frac{4}{3}$, which is already what we wished to prove. Recall, moreover, that this implies $L^r(\mathbb{R}^2) \to L^q(\mathbb{S}^1)$ estimates in the optimal two-dimensional restriction range $1 \leq r < \frac{4}{3}, r' \geq 3q$.

$\square$

3. The high-dimensional result

Just like we employed our techniques to deal with the two-dimensional case, we adapt the arguments by M. Vitturi [11] to achieve high-dimensional estimates. We briefly sketch on how to do it.

Theorem 4. *Let $d \geq 3$, and*

$$\mathfrak{M}f(x) = \sup_{0<\varepsilon\leq 1} \fint_{B(0,\varepsilon)} |\widehat{f}(x+y)| \mathrm{d}y.$$

Then it holds that

$$\|\mathfrak{M}f\|_{L^q(\mathbb{S}^{d-1})} \leq C_{p,q,d}\|f\|_{L^p(\mathbb{R}^d)},$$

where $1 \leq p \leq \frac{4}{3}$ and $p' \geq \frac{d+1}{d-1}q$.

Proof. First, write the auxiliary bilinear operator

$$\mathfrak{M}(f;g)(x) = \sup_{0\leq\varepsilon\leq 1} \left| \fint_{B(0,\varepsilon)} \widehat{f}(x+y)g(x+y)\, \mathrm{d}y \right|.$$

Letting $\mathfrak{A}_{\varepsilon(\cdot),g}f(x) = \fint_{B(0,\varepsilon(x))} \widehat{f}(x+y)g(x+y)\,\mathrm{d}y$ be the linearised operator for suitable measurable g, ε, $\|g\|_\infty = 1$. Its adjoint has the form

$$\mathfrak{A}^*_{\varepsilon(\cdot),g}h(\xi) = \int_{\mathbb{S}^{d-1}} G(x,\xi)e^{2\pi i\xi\cdot x}h(x)\,\mathrm{d}\sigma(x),$$

σ standing for the surface measure on the $(d-1)-$dimensional sphere, and $G(x,\xi) = \mathcal{F}(g(x+\cdot)\chi_{B(0,\varepsilon(x))})(\xi)$. Following Vitturi's arguments and the ones in the proof of Theorem 2, it is enough to prove the following estimate:

$$\|\mathfrak{A}^*_{\varepsilon(\cdot),g}h\|_{L^4(\mathbb{R}^d)} \leq C_{q,d}\|h\|_{L^{q_d'}(\mathbb{S}^{d-1})},$$

where $q_d = 4\frac{d+1}{d-1}$. Now we write the L^4 norm as a (square root of a) L^2 norm of the convolution of the Fourier transform $(\mathfrak{A}^*_{\varepsilon(\cdot),g}h)\widehat{\ }$ with itself. With this in mind, one gets from a calculation that

$$(\mathfrak{A}^*_{\varepsilon(\cdot),g}h)\widehat{\ }(\eta) = g(\eta)\int_{\mathbb{S}^{d-1}} h(x)\chi_{B(0,\varepsilon(x))}(\eta - x)\mathrm{d}x =: g(\eta)T_{\varepsilon(\cdot)}h(\eta).$$

We are then able to bound

$$|(\mathfrak{A}^*_{\varepsilon(\cdot),g}h)\widehat{\ } * (\mathfrak{A}^*_{\varepsilon(\cdot),g}h)\widehat{\ }(\rho)| \leq |(T_{\varepsilon(\cdot)}|h|) * (T_{\varepsilon(\cdot)}|h|)(\rho)|.$$

But the operator on the right hand side has been already treated in Vitturi's proof, and therefore we can conclude the desired bounds from the ones in [11]. $\square$

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