Stable vectors in the Moy-Prasad filtration

Jessica Fintzen (Harvard University)

Large goal / motivation

Understand and classify all complex representations of p-adic groups.
$\rightsquigarrow$ Local Langlands correspondence

What are p-adic numbers?

In number theory the study of congruences plays an important role, and hence we want to define a norm on the integers so that two numbers n and m are close iff $n \cong m \mod p^N$ for a large N. This is achieved by

$$|p^s \cdot r| = p^{-s} \text{ with } r \text{ and } p \text{ coprime.}$$

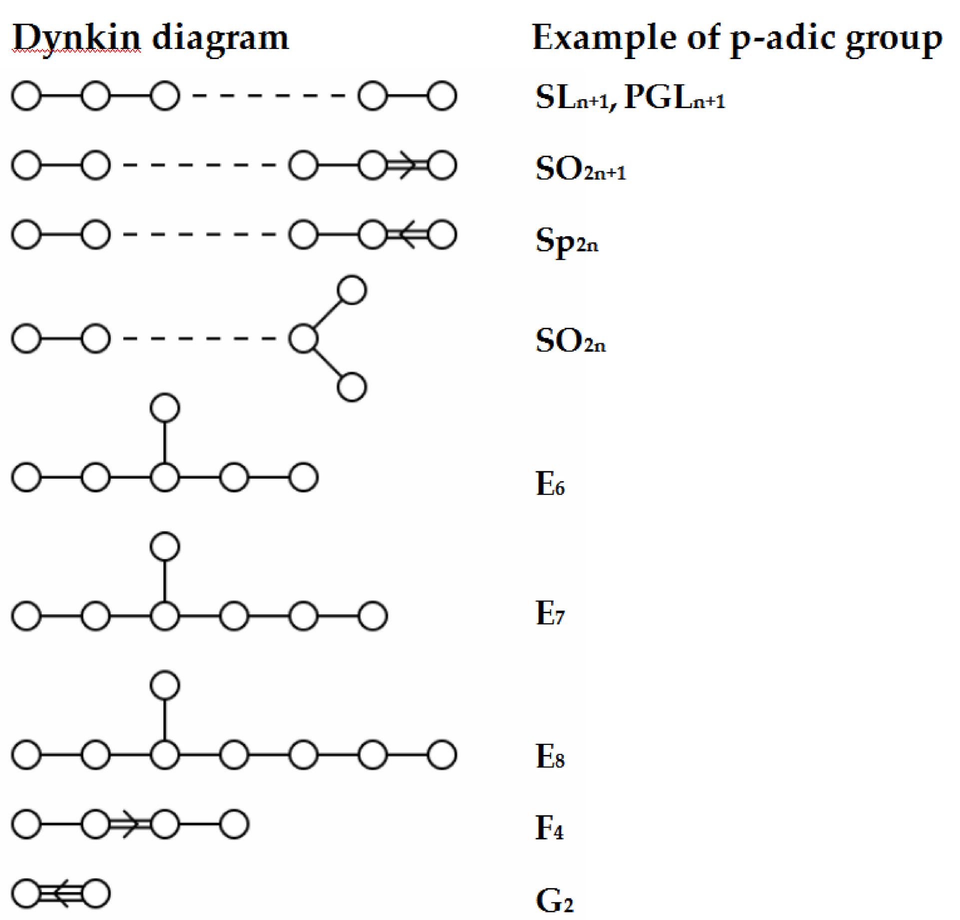
The p-adic integers $\mathbb{Z}_p$ are the completion of the integers by this norm $|\cdot|$, i.e. a p-adic integer is of the form

$$a_0 + a_1 \cdot p + a_2 \cdot p^2 + a_3 \cdot p^3 + \ldots \text{ for some integers } a_i.$$

The p-adic numbers $\mathbb{Q}_p$ are the fraction field of the p-adic integers. They are a completion of the rational numbers $\mathbb{Q}$.
We call a field F that is a finite extension of the p-adic numbers $\mathbb{Q}_p$ **a p-adic field**.

What are p-adic groups?

P-adic groups, or more precisely, **reductive groups over p-adic fields**, are certain subgroups of the group of invertible n by n matrices whose entries are elements of a p-adic field F, e.g. $GL_n(F), SL_n(F), SO_n(F), Sp_n(F)$. On this poster, we will restrict our attention to the simple factors of these groups, and we make the assumption that our group is split - a technical assumption that is always satisfied for reductive groups over algebraically closed fields. These simple split groups are classified up to a finite center in terms of a combinatorial object, the Dynkin diagram.



Moy-Prasad filtration

The Bruhat-Tits building $\mathscr{B}(G, F)$

- is a building associated to a given p-adic group G by Bruhat and Tits
- for $SL_2(\mathbb{Q}_p)$ it is an infinite tree in which each vertex has $p+1$ neighbors, see Figure 1
- for every point x in the building, Bruhat and Tits define a compact subgroup G_x in G, called the **parahoric subgroup**, which has finite index in the stabilizer $\mathrm{Stab}_G(x)$

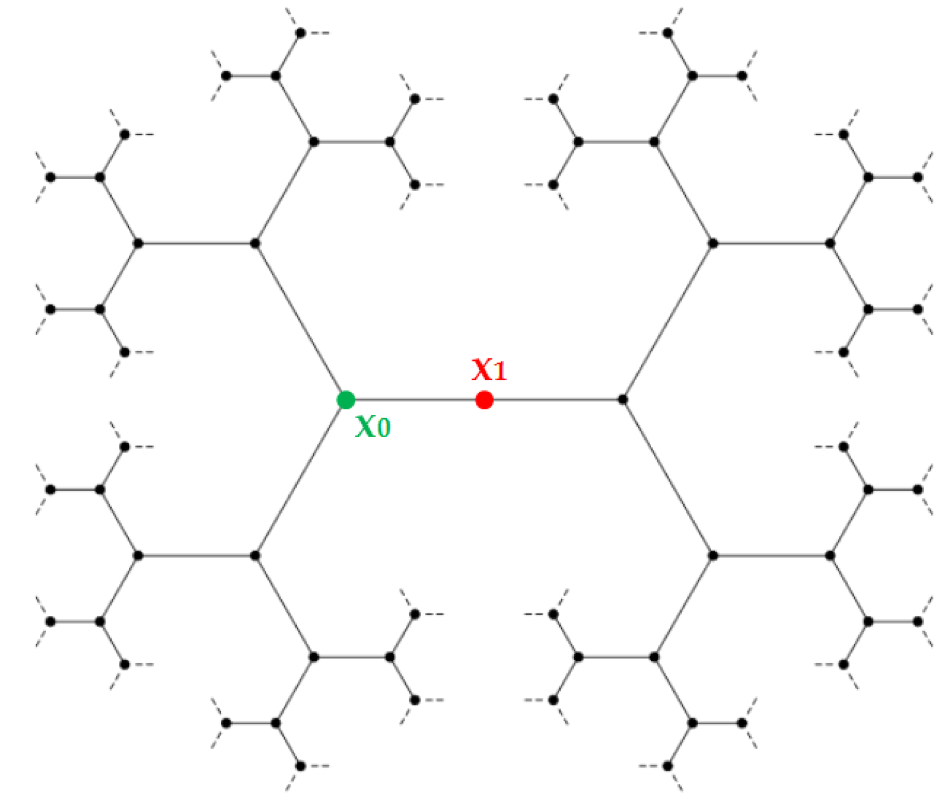


Figure 1: Bruhat-Tits building of $SL_2(\mathbb{Q}_2)$; source: [Rab05].

The Moy-Prasad filtration generalizes the filtrations

$$\mathbb{Z}_p \supset p \cdot \mathbb{Z}_p \supset p^2 \cdot \mathbb{Z}_p \supset p^3 \cdot \mathbb{Z}_p \supset \ldots$$
$$\text{and} \quad \mathbb{Z}_p^\times \supset 1 + p \cdot \mathbb{Z}_p \supset 1 + p^2 \cdot \mathbb{Z}_p \supset \ldots$$

to arbitrary parahoric subgroups G_x of a p-adic group.

Examples for $G = SL_2(\mathbb{Q}_2)$, where $\mathfrak{p} = 2 \cdot \mathbb{Z}_2$ (see Figure 1):

$$G_{x_0} = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix} \qquad G_{x_1} = \begin{pmatrix} \mathbb{Z}_2 & \mathfrak{p} \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$$

$$G_{x_1,0.5} = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathbb{Z}_2 & 1+\mathfrak{p} \end{pmatrix}$$

$$G_{x_0,1} = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \qquad G_{x_1,1} = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p}^2 \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$$

$$G_{x_1,1.5} = \begin{pmatrix} 1+\mathfrak{p}^2 & \mathfrak{p}^2 \\ \mathfrak{p} & 1+\mathfrak{p}^2 \end{pmatrix}$$

$$G_{x_0,2} = \begin{pmatrix} 1+\mathfrak{p}^2 & \mathfrak{p}^2 \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{pmatrix} \qquad G_{x_1,2} = \begin{pmatrix} 1+\mathfrak{p}^2 & \mathfrak{p}^3 \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{pmatrix}$$

Important property: The quotient of the parahoric subgroup by the first proper filtration subgroup, called the **reductive quotient**, acts on all other quotients of subsequent Moy-Prasad filtration groups, and the latter are isomorphic to vector spaces over a field of characteristic p.

Example: For $SL_2(\mathbb{Q}_2)$, $G_{x_0}/G_{x_0,1} \simeq SL_2(\mathbb{F}_2)$ acts on $G_{x_0,1}/G_{x_0,2} \simeq \mathrm{Mat}_{2\times 2}(\mathbb{F}_2)_{\text{trace}=0}$ via conjugation.

Stable vectors and epipelagic representations

Supercuspidal representations

- are the building blocks for all representation of p-adic groups
- very mysterious, only few constructions known, see [Adl98] (special case) and [Yu01] (for large p)

Epipelagic representations are supercuspidal representations of smallest positive depth.

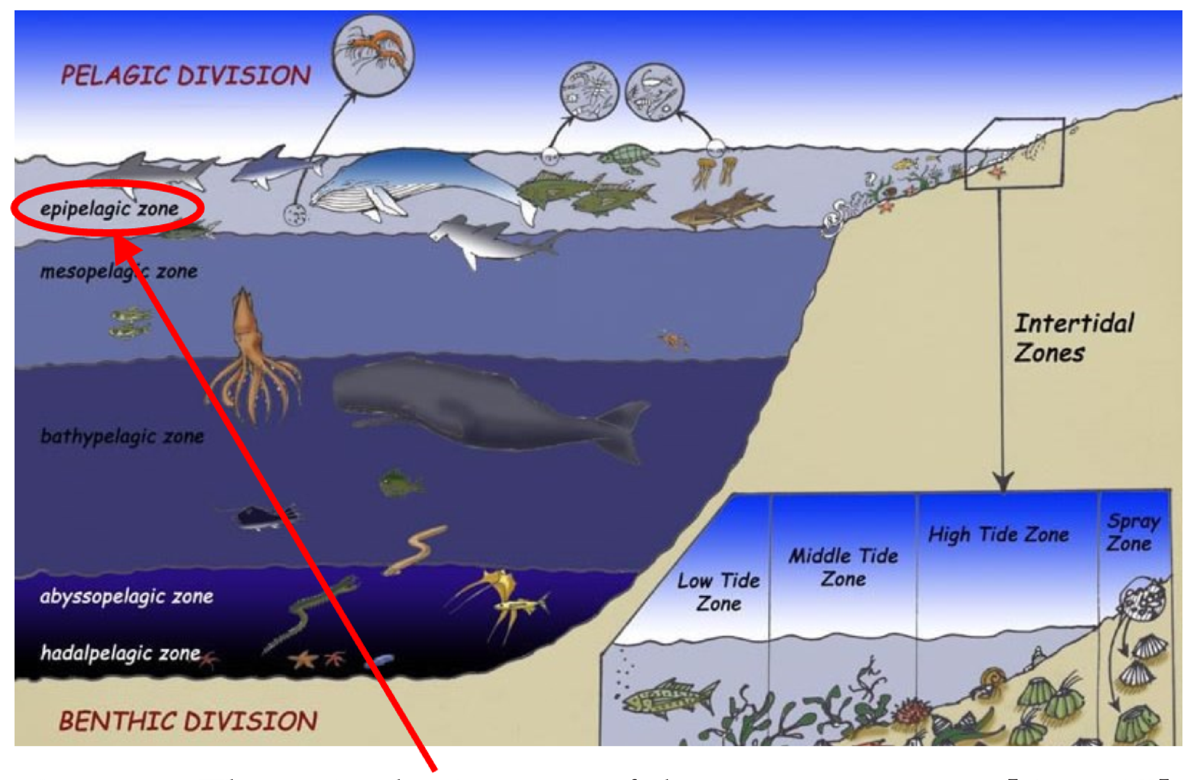
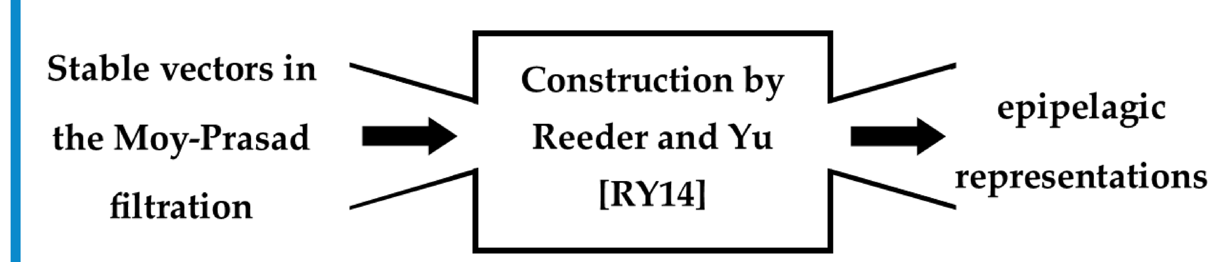


Figure 2: The epipelagic zone of the ocean; source: [Ams15].

A **stable vector** is a vector v in some representation V of a reductive group G over an algebraically closed field whose stabilizer $\mathrm{Stab}_G(v)$ is finite and whose G-orbit is closed in the Zariski topology.
Given a stable vector in the dual of the first Moy-Prasad filtration quotient, Reeder and Yu gave recently a construction of epipelagic representations.



This construction works uniformly for all primes p, but it requires the existence of stable vectors. Reeder and Yu gave a necessary and sufficient criterion for the existence of stable vectors only for large primes p.

Main Theorem

(joint with Beth Romano)
Theorem 1 (in words). *The existence of stable vectors in the Moy-Prasad filtration quotient does not depend on the prime p.*
Theorem 1 (details for experts). *Let $x \in \mathscr{B}(G, F^{ur})$ be a rational point of order m. Then there exist stable vectors in $\left(G_{x,\frac{1}{m}}/G_{x,\frac{1}{m}+}\right)^\vee$ under the action of $G_{x,0}/G_{x,0+}$ if and only if there exists an elliptic, $\mathbb{Z}$-regular element of order m in the Weyl group of G and x is conjugate to $x_0 + \frac{1}{m}\check{\rho}$ under the affine Weyl group for some hyperspecial point x_0. Here $\check{\rho}$ is half of the sum of the positive co-roots.*

This theorem was known for large primes p thanks to [RY14].

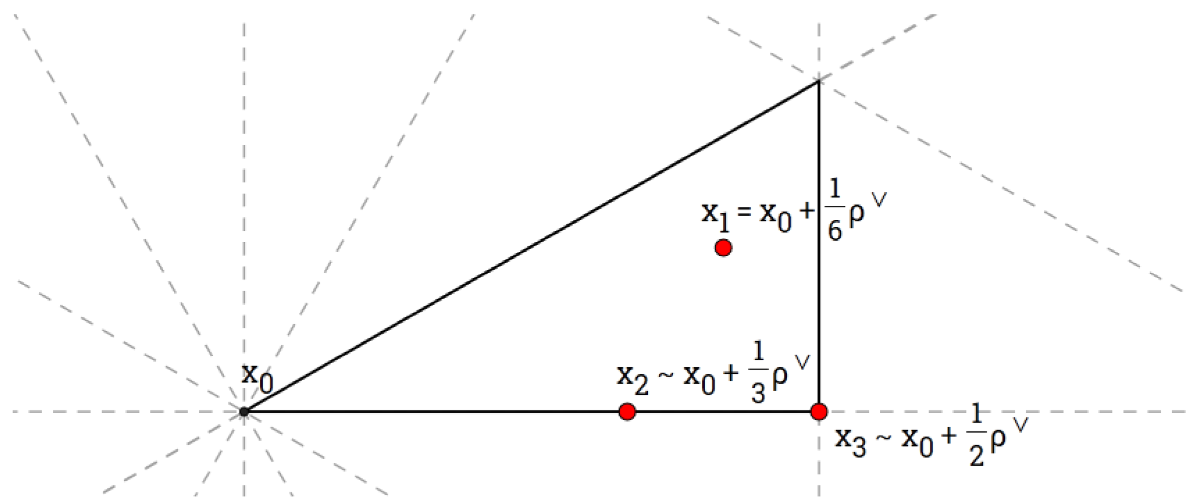


Figure 3: A fundamental chamber for G_2 with all points (in red) for which there exist stable vectors in the dual of the first Moy-Prasad filtration quotient - independent of p!

Applications

- We obtain supercuspidal (epipelagic) representations uniformly for all primes p.
- As a corollary of the proof we obtain a different description of the Moy-Prasad filtration quotient as a representation of the reductive quotient for all primes p without restriction.
- The proof involves a construction of the filtration quotient representations over the integers. As a consequence we can compare the occuring representations of the reductive quotients at different primes.

References

[Adl98] Jeffrey D. Adler. Refined anisotropic K-types and supercuspidal representations. *Pacific J. Math.*, 185(1):1–32, 1998.

[Ams15] Sheri Amsel. Glossary (What Words Mean) with Pictures!. 2005-2015. April 2, 2015, `http://www.exploringnature.org/db/detail.php?dbID=13&detID=406`.

[Rab05] Joseph Rabinoff. The Bruhat-Tits building of a p-adic Chevalley group and an application to representation theory. 2005. available at `http://people.math.gatech.edu/~jrabinoff6/papers/building.pdf`.

[RY14] Mark Reeder and Jiu-Kang Yu. Epipelagic representations and invariant theory. *J. Amer. Math. Soc.*, 27(2):437–477, 2014.

[Yu01] Jiu-Kang Yu. Construction of tame supercuspidal representations. *J. Amer. Math. Soc.*, 14(3):579–622 (electronic), 2001.