ADVANCED TOPICS IN ELLIPTIC REGULARITY

FRANZ GMEINEDER

Background Information

These are informal notes for my lecture on advanced topics in elliptic regularity (V5B5), summer term 2019, at the University of Bonn. The aim of the course is to outline several recent trends and research areas within the broad field of elliptic regularity.

The main prerequisites are as follows:

- Measure and integration theory (Analysis 1–3),
- Functional analysis (especially, familiarity with weak convergence, the Banach-Alaoglu theorem, Sobolev spaces),
- elementary elliptic partial differential equations (especially, the Laplace- and Poisson equations)

Updated lecture notes and background material will be posted after each lecture on the course webpage

`https://www.math.uni-bonn.de/ag/ana/SoSe2019/V5B5_SoSe_19`

The course takes place

- Mondays, 4–6pm (c.t.) in SR 1.007 (Endenicher Allee 60),
- Thurdays, 4–6pm (c.t.) in SR 1.008 (Endenicher Allee 60).

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1. Introduction

A variety of partial differential equations can be recast as variational problems. Minimisers of the corresponding variational integrals then provide (weak) solutions. On the other hand, several physical problems directly lead to minimum problems, and this course intends to outline ways how to approach the regularity of the corresponding minima. In doing so, we shall survey old and new questions and techniques in the field, aiming to make connections with problems from applications such as physics or geometry. This introductory chapter serves to set up a unifying theme for such problems, and giving some ideas what the course shall be about.

To provide a overarching framework for variational problems, the present course is mainly centered around functionals

$$\mathscr{F}[v;\Omega] := \int_{\Omega} F(x,v,\nabla v)\,\mathrm{d}x, \tag{1.1}$$

where $F\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to\mathbb{R}$ is a variational integrand and Ω is an open and bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. Accordingly, the variational problem of major interest in this course is

$$\text{to minimise } \mathscr{F}[-;\Omega] \;\text{ over a class of competitors } X(\Omega;\mathbb{R}^N). \tag{1.2}$$

Typically, $X(\Omega;\mathbb{R}^N)$ shall be a Dirichlet subclass of some Sobolev space $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$. We will recap the elementary properties of such spaces below, but for the time being it suffices to recall that a map $v\in\mathrm{L}^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ belongs to $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ if and only if $v\in\mathrm{L}^p(\Omega;\mathbb{R}^N)$ and its weak gradient Dv belongs to $\mathrm{L}^p(\Omega;\mathbb{R}^{N\times n})$. Moreover,

$$\|v\|_{\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)} := \Big(\|v\|^p_{\mathrm{L}^p(\Omega;\mathbb{R}^N)} + \|Dv\|^p_{\mathrm{L}^p(\Omega;\mathbb{R}^N)}\Big)^{\frac{1}{p}}, \qquad v\in\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N),$$

is a norm on $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ which makes the latter a Banach space. It is also convenient to introduce $\mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)$ as the closure of $\mathrm{C}^\infty_c(\Omega;\mathbb{R}^N)$ with respect to $\|\cdot\|_{\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)}$. We will augment auxiliary facts on weakly differentiable functions as the course evolves.

For simplicity and to give a quick outline of the chief obstructions in view of (1.2), we firstly suppose that

(i) F is *autonomous* (and thus is independent of the first two variables):

$$F(x,y,z) = F(z),$$

(ii) F is *of p-growth*, that is, there exist $c_1,c_2,c_3>0$ such that

$$c_1|z|^p - c_2 \leqslant F(z) \leqslant c_3(1+|z|^p) \qquad \text{for all } z\in\mathbb{R}^{N\times n}. \tag{p-growth}$$

In view of condition (ii) below, the variational integral (1.1) is well-defined on any subset of $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$. To incorporate Dirichlet constraints, we let $u_0\in\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ be a *Dirichlet datum* and consider the

$$\text{minimisation of } \mathscr{F}[v] := \int_\Omega F(Dv)\,\mathrm{d}x \qquad \text{over } X(\Omega;\mathbb{R}^N) := \mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N), \tag{1.3}$$

where $\mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N) := u_0 + \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)$. Our first overall objective is to ensure the existence of *minima*, and so we start by making the following

Definition 1.1 (Minimisers/local minimiser)**.** *Let u_0 and F be as above. We then say*

(a) $v \in \mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)$ *is a* minimiser *if and only if there holds*

$$\mathscr{F}[v;\Omega] \leqslant \mathscr{F}[v+\varphi;\Omega] \qquad \textit{for all } \varphi \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N).$$

(b) $v \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *is a* local minimiser *if and only if for any Lipschitz subset* $\omega \Subset \Omega$ *there holds*

$$\mathscr{F}[v;\omega] \leqslant \mathscr{F}[v+\varphi;\omega] \qquad \textit{for all } \varphi \in \mathrm{W}^{1,p}_0(\omega;\mathbb{R}^N).$$

The notion of local minimality is of secondary importance by now but is included for completeness here; later on, it provides a suitably flexible notion for stating regularity results. Assuming a minimiser $u \in \mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)$, we briefly connect to the introductory theme: If $F \in \mathrm{C}^1(\mathbb{R}^{N\times n})$, then minimality of u readily yields

$$\int_\Omega \frac{F(u\pm\varepsilon\varphi)-F(u)}{\varepsilon}\,\mathrm{d}x \geqslant 0 \qquad \text{for all } \varphi \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N).$$

In this situation, sending $\varepsilon \searrow 0$ yields the corresponding *Euler-Lagrange system*

$$\int_\Omega \langle F'(Dv), D\varphi\rangle\,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)$$

whenever the expression on the left-hand side makes sense. We can read this equation or system of equations, respectively, as a *weak variant* of the partial differential equation

$$\begin{cases} -\operatorname{div}(F'(\nabla u)) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

We turn to some examples.

Example 1.2 (The p-Laplacean equation)**.** *Our first example of an integrand that falls into the realm of the above theory is that of the p-Laplacean operator, $F(z) := \frac{1}{p}|z|^p$. If $1 < p < \infty$, then $F \in \mathrm{C}^1(\mathbb{R}^{N\times n})$, and the corresponding Euler-Lagrange system reads as*

$$\int_\Omega |Dv|^{p-2}\langle Dv, D\varphi\rangle\,\mathrm{d}x = 0 \qquad \textit{for all } \varphi \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N). \tag{1.5}$$

Assuming sufficient regularity of the map v, we may view the last equation as the weak formulation of the p-Laplacean equation

$$-\Delta_p u := -\operatorname{div}(|Du|^{p-2}Du) = 0 \qquad \textit{in } \Omega \tag{1.6}$$

subject to the Dirichlet constraint $u = u_0$ on $\partial\Omega$. We note that setting $p = 2$ lets us retrieve the usual homogeneous Laplacean equation $\Delta u = 0$. If $1 < p < n$, then we have $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}^{\frac{np}{n-p}}(\Omega;\mathbb{R}^N)$ by the Sobolev embedding theorem. Hence, letting $q := \frac{np}{np-n+p}$ and picking some $g \in \mathrm{L}^q(\Omega;\mathbb{R}^N)$, we put $\widetilde{F}(x,y,z) := F(z) - \langle g(x), y\rangle$. Supposing that a minimiser u exists for the variational principle (1.2) *with F replaced by $\widetilde{F}$, then the Euler-Lagrange equation reads as*

$$\int_\Omega \langle F'(Du), D\varphi\rangle - \langle g,\varphi\rangle\,\mathrm{d}x = 0 \qquad \textit{for all } \varphi \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N).$$

Recalling that $F(z) = \frac{1}{p}|z|^p$, the latter can be read as the weak version of the inhomogeneous p-Laplacean equation

$$-\operatorname{div}(|Du|^{p-2}Du) = g \qquad \textit{in } \Omega.$$

In the previous example, we excluded $p = 1$ as $z \mapsto |z|$ is not of class C^1. We shall turn to this case later on in the course of the text, however, this requires more preparations. Instead, we consider

Example 1.3 (The minimal surface equation)**.** *An integrand that matches condition (iii) from above with $p = 1$ is that of the* minimal surface integrand*,*

$$F(z) := \sqrt{1 + |z|^2}, \qquad z \in \mathbb{R}^{N\times n}.$$

The corresponding Euler-Lagrange system then reads as

$$\begin{cases} H(u) := -\operatorname{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

The operator H is sometimes referred to as the mean curvature operator*.*

1.1. **The direct method.** We now proceed by establishing the existence of minima for the sample Dirichlet problem (1.2) by means of the *direct method of the calculus of variations*. To this end, we make use of some machinery from functional analysis to be recalled step by step below.

First, we must ensure that $\mathscr{F}$ is bounded below on the class $\mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)$. Subject to ($p$-growth), we then estimate

$$-\infty < -c_2 \leqslant c_1 \int_\Omega |Dv|^p \,\mathrm{d}x - c_2 \leqslant \mathscr{F}[v] \qquad \text{for all } v \in \mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N).$$

Therefore, $m := \inf \mathscr{F}[\mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)]$ exists in $\mathbb{R}$. We may thus pick a *minimising sequence* $(u_j) \subset \mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)$, that is,

$$\mathscr{F}[u_j] \longrightarrow m, \quad j \to \infty.$$

Our next aims are to demonstrate that (u_j) converges in a suitable sense to some $u \in \mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)$ and that this map u is a minimiser of $\mathscr{F}$. For the first objective, we must lift the information available for the gradients to an information about the competitor maps per se. To do so, we first recall the *Poincaré inequality on* $\mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)$:

Lemma 1.4 (Poincaré inequality on $\mathrm{W}^{1,p}_0$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then, for any $1 \leqslant p < \infty$, there exists $c = c(p,n) > 0$ such that*

$$\|v\|_{\mathrm{L}^p(\Omega;\mathbb{R}^N)} \leqslant c\|Dv\|_{\mathrm{L}^p(\Omega;\mathbb{R}^{N\times n})}$$

holds for all $v \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)$.

Based on this lemma, we establish the boundedness of minimising sequences as follows: Let $v \in \mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)$ so that $v = u_0 + \psi$ for some $\psi \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)$. Then, denoting the constant appearing in the preceding lemma by $c_{\mathrm{Poinc},p} > 0$,

$$\begin{aligned} \|v\|^p_{\mathrm{W}_{1,p}(\Omega;\mathbb{R}^N)} &\leqslant \|v\|^p_{\mathrm{L}^p(\Omega;\mathbb{R}^N)} + \|Dv\|^p_{\mathrm{L}^p(\Omega;\mathbb{R}^{N\times n})} \\ &\leqslant (c^p_{\mathrm{Poinc},p} + 1)\|Dv\|_{\mathrm{L}^p(\Omega;\mathbb{R}^{N\times n})} \\ &\leqslant \frac{c^p_{\mathrm{Poinc},p} + 1}{c_1}\Big(\int_\Omega F(Dv)\,\mathrm{d}x + c_2\mathscr{L}^n(\Omega)\Big). \end{aligned}$$

In consequence, inserting the minimising sequence from above, we conclude

$$\sup_{j\in\mathbb{N}} \|u_j\|_{\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)} \leqslant C < \infty.$$

Thus, (u_j) is bounded in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$. At this stage, we specify to

$$\boxed{1<p<\infty,}$$

a device whose importance becomes clear by the next

Lemma 1.5 (Banach-Alaoglu-Bourbaki)**.** *Let $(X,\|\cdot\|)$ be a reflexive, real Banach space. Suppose that $(x_j)\subset X$ is a bounded sequence in X. Then there exists a subsequence $(x_{j(i)})\subset(x_j)$ and some $x\in X$ such that $x_{j(i)}\rightharpoonup x$ in X as $i\to\infty$.*

If $1<p<\infty$, then $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ *is reflexive* and thus Lemma 1.5 is available. As a consequence, we find $u\in\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ and a subsequence $(u_{j(i)})\subset(u_j)$ such that $u_{j(i)}\rightharpoonup u$ in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$. Toward the minimality of u, we now ensure that u is an admissible competitor map for the present variational principle. In fact, we have $u\in\mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)$ as a consequence of the next

Lemma 1.6 (Trace theorem for $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$)**.** *Let $\Omega\subset\mathbb{R}^N$ be an open and bounded Lipschitz domain. Then for any $1<p<\infty$ there exists a bounded linear operator*

$$\mathrm{Tr}\colon\ \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)\to\mathrm{L}^p(\partial\Omega;\mathbb{R}^N)$$

which satisfies $\mathrm{Tr}(v)=v|_{\partial\Omega}$ *for all* $v\in\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)\cap\mathrm{C}(\overline{\Omega};\mathbb{R}^N)$*. Moreover,* Tr *is continuous for weak convergence on* $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$*, and we have*

$$\mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)=\ker(\mathrm{Tr}).$$

Working from the previous lemma, $u_{j(i)}\rightharpoonup u$ in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ readily yields that $\mathrm{Tr}(u)=u_0$ and thus $u\in\mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)$. To conclude the existence proof, we turn to the actual minimality of u.

Definition 1.7 (Lower semicontinuity)**.** *Let X be a real vector space and '$\rightsquigarrow$' a notion of sequential convergence on X. We say that $T\colon X\to\mathbb{R}$ is* lower semicontinuous with respect to '$\rightsquigarrow$' *on X provided there holds*

$$T(x)\leqslant\liminf_{j\to\infty}T(x_j)$$

whenever $x,x_1,x_2,...\in X$ are such that $x_j\rightsquigarrow x$ as $j\to\infty$.

Lower semicontinuity can be put into a much more general fashion in the framework of topological spaces, but we do not need this in the sequel. In the particular situation where $X=\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$, the requisite notion of convergence '$\rightsquigarrow$' clearly is that of weak convergence, and we say that $T\colon\ \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)\to\mathbb{R}$ is *sequentially weakly lower semicontinuous* (SWLSC) if T is lower semicontinuous for sequential weak convergence on $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$.

Now *suppose that* $\mathscr{F}$ given by (1.2) *is* SWLSC on $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$. Then we deduce from $u_{j(i)}\rightharpoonup u$ in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ that

$$\mathscr{F}[u]\leqslant\liminf_{i\to\infty}\mathscr{F}[u_{j(i)}]=m=\inf\mathscr{F}[\mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)].$$

On the other hand, $u\in\mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)$ and therefore

$$\mathscr{F}[u]\leqslant\inf\mathscr{F}[\mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)]\leqslant\mathscr{F}[u]\Rightarrow\mathscr{F}[u]=\inf\mathscr{F}[\mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)].$$

Thus, $u\in\mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)$ is a minimiser of $\mathscr{F}$.

To conclude the existence proof, we thus require a condition on F that makes the integral functional (1.2) lower semicontinuous with respect to sequential weak convergence on $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$. Such a condition is exemplarily given by *convexity* of F, i.e.,

$$F(\lambda z_1 + (1-\lambda)z_2) \leqslant \lambda F(z_1) + (1-\lambda)F(z_2)$$

for all $\lambda \in [0,1]$ and all $z_1, z_2 \in \mathbb{R}^{N\times n}$. In general – so, e.g., when $N > 1$ – this condition however is *not necessary* for the requisite lower semicontinuity assertion. The correct substitute then is given by MORREY's notion of *quasiconvexity*:

Definition 1.8 (Quasiconvexity)**.** *Let $F \in \mathrm{C}(\mathbb{R}^{N\times n})$. We say that F is* quasiconvex *provided there holds*

$$F(z) \leqslant \int_Q F(z + D\varphi)\,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{W}_0^{1,\infty}(\Omega;\mathbb{R}^N),$$

where $Q = (0,1)^n$ is the open unit cube in $\mathbb{R}^n$.

Some comments are in order. First, quasiconvexity generalises convexity. Indeed, if $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is convex, then we have by Jensen's inequality

$$F(z) = F\Big(\int_Q z + D\varphi\,\mathrm{d}x\Big) \leqslant \int_Q F(z + D\varphi)\,\mathrm{d}x$$

for all $z \in \mathbb{R}^{N\times n}$ and all $\varphi \in \mathrm{W}_0^{1,\infty}(\Omega;\mathbb{R}^N)$. Also, if $F \in \mathrm{C}(\mathbb{R}^{N\times n})$ is quasiconvex, then we have

$$F(z) \leqslant \fint_\Omega F(z + D\varphi)\,\mathrm{d}x$$

for any open set $\Omega \subset \mathbb{R}^n$, $z \in \mathbb{R}^{N\times n}$ and $\varphi \in \mathrm{W}_0^{1,\infty}(\Omega;\mathbb{R}^N)$. From here we obtain the more geometric interpretation that quasiconvex functionals are locally minimised by affine-linear maps $v(x) = Ax + b$, where $A \in \mathbb{R}^{N\times n}$ and $b \in \mathbb{R}^n$. Whereas the above shows that convexity implies quasiconvexity, the converse is not true in general (except for some specific dimensional constellations); e.g., the determinant det *is* quasiconvex, however, clearly non-convex. To conclude, we now come to the statement that links quasiconvexity with the requisite lower semicontinuity:

Theorem 1.9 (Lower semicontinuity)**.** *Let $1 \leqslant p < \infty$ and suppose that $F \in \mathrm{C}(\mathbb{R}^{N\times n})$ satisfies*

(a) *$0 \leqslant F(z) \leqslant c(1 + |z|^p)$ for some constant $c > 0$ and all $z \in \mathbb{R}^{N\times n}$,*
(b) *the quasiconvexity condition from Definition 1.8.*

Then the correspnding integral functional

$$\mathscr{F}[v] := \int_\Omega F(Dv)\,\mathrm{d}x$$

is sequentially weakly lower semicontinuous on $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$.

We shall give an outline of the proof at some later stage of the course. By now, we confine to stating that if $F \in \mathrm{C}(\mathbb{R}^{N\times n})$ is

- is quasiconvex,
- satisfies (ii) and (iii) from above for some $1 < p < \infty$ and
- $u_0 \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$,

then the variational principle

$$\text{to minimise}\quad \mathscr{F}[v] := \int_{\Omega} F(Dv)\,\mathrm{d}x \quad \text{over}\quad \mathrm{W}^{1,p}_{u_0}(\Omega;\mathbb{R}^N)$$

has a solution.

Rather than embarking on a detailled description of the case $p = 1$ – which we omitted completely in the above discussion and which we shall return to in detail as the course evolves – we confine to gathering the main obstructions in comparison with the superlinear, i.e., $1 < p < \infty$-growth case.

(a) Whereas $\mathscr{F}$ is bounded below on $\mathrm{W}^{1,1}_{u_0}(\Omega;\mathbb{R}^N)$ and minimising sequences are bounded in $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$, the latter space is *not reflexive*. As a consequence, we cannot use the Banach-Alaoglu-Bourbaki theorem in the above form. On an explicit level, the mere L^1-control on the gradients does not rule out concentration effects.

(b) The general problem setup of (a) suggests that the gradients of the single members of minimising sequences might concentrate and, in a suitable sense, converge to measures. However, as the functional $\mathscr{F}$ is initially defined for gradients being L^p-maps, it is far from clear how to make sense of (1.1) when being applied to gradients which are measures.

(c) As shall turn out, it *is possible* to set up a function space framework within which a relaxed version of $\mathscr{F}$ can be dealt with. In this setting, it is however not possible to ensure that the minimiser provided by means of the direct method *belongs to the prescribed Dirichlet class*[1].

The detailled discussion and resolution of these issues will lead us to the space of *functions of bounded variation* $\mathrm{BV}(\Omega;\mathbb{R}^N)$.

1.2. **Regularity and function spaces.** In the preceding subsection we have established the existence of minimisers in a rather general framework. Broadly speaking, the aim of *regularity theory* is to inquire whether

$$\boxed{\textit{minima are genuinely better behaved than generic competitor maps.}}$$

Here, the notion of 'better' is a matter of taste, and we proceed by formalising this issue by use of embedding theorems.

By definition of the Dirichlet classes involved, minima belong to the Sobolev space $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$. Elements of the latter space possess some inherent regularity. In fact, by the usual embedding theorems we exemplarily have that

$$\begin{aligned}
&1 \leqslant p < n && \Longrightarrow \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N) \hookrightarrow \mathrm{L}^{\frac{np}{n-p}}(\Omega;\mathbb{R}^N),\\
&p = n && \Longrightarrow \mathrm{W}^{1,n}(\Omega;\mathbb{R}^N) \hookrightarrow \mathrm{BMO}(\Omega;\mathbb{R}^N),\\
&n < p < \infty && \Longrightarrow \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N) \hookrightarrow \mathrm{C}^{0,1-\frac{n}{p}}(\Omega;\mathbb{R}^N).
\end{aligned}$$

As such, (higher) regularity of a minimiser or a (weak) solution of a partial differential equation can be viewed as *any property that goes beyond embedding results available for* all *elements of the competitor class.*

Regularity theory thus can be viewed as the precise investigation of the properties of minima (or solutions of partial differential equations). This research area has a long history and rich tradition; most notably, it occurs in HILBERT's 23(+1)

[1]Essentially because the corresponding trace operator is not continuous for the sort of convergence that yields the requisite compactness.

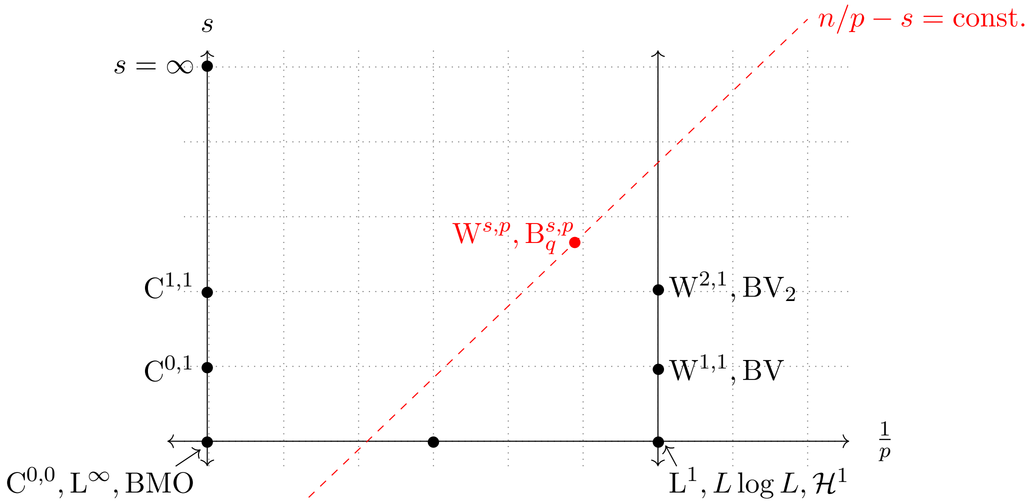

FIGURE 1. Function Space Diagram; schematic presentation of function spaces. The parameter p displays the integrability parameter and s displays the smoothness parameter.

problems posed at the International Congress of Mathematicians in 1900, reading as follows:

> HILBERT'S 19TH PROBLEM: *Sind die Lösungen regulärer Variationsprobleme stets notwendig analytisch?* (*Are solutions of regular variational problems necessarily analytic?*)

An analogous and related question posed at the same occasion reads as this:

> HILBERT'S 20TH PROBLEM: ... *ob nicht jedes reguläre Variationsproblem eine Lösung besitzt, sobald hinsichtlich der gegebenen Grenzbedingungen gewisse Annahmen etwa die Stetigkeit und stückweise öftere Differenzierbarkeit der für die Randbedingungen maßgebenden Funktionen erfüllt sind und nötigenfalls der Begriff der Lösung eine sinngemäße Erweiterung erfährt.* (*...whether or not every regular variational problem does possess a solution, as long as certain – in view of the boundary conditions – assumptions are made, so for instance continuity and piecewise higher differentiability, and the notion of solution is possibly adapted accordingly.)*)

In order to systematise the question of regularity, it is suitable to get an impression of which function spaces a given competitor class embeds into *or not, respectively.* Since there is an abundance in possible function spaces to be utilised to state regularity results, we first attempt to get an *overview* of possible relevant properties. Here we shall make our way through the following diagram as the course evolves, commonly understood as a *function space* or *adaptivity diagram.* We now exemplarily discuss selected regularity questions for the well-known Laplace or Poisson equations, involving the

• C^k*-scale.* Suppose that $\mathbb{A} \in \mathbb{S}(\mathbb{R}^{N\times n})$ is an elliptic bilinear form, so that there exists $\lambda > 0$ such that

$$\mathbb{A}[\xi, \xi] \geqslant \lambda|\xi|^2.$$

We then say that $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is locally $\mathbb{A}$-harmonic provided for all $\omega \Subset \Omega$ there holds

$$\int_\omega \mathbb{A}[Du, D\varphi]\,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{W}^{1,2}_0(\omega;\mathbb{R}^N).$$

As a consequence of the difference quotient characterisation of Sobolev maps $v \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ – cf. Lemma 2.1 for the more general context of exponents $1 < p < \infty$ instead of $p = 2$ – we then infer that $u \in \mathrm{C}^\infty$: Localising and inserting test maps $\varphi = \Delta^-_{s,h}(\rho^2\Delta_{s,h}(u-a)$, we firstly find $\varphi \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^N)$ so that φ is admissible as a test map in the definition of $\mathbb{A}$-harmonicity. We then obtain

$$\int_\Omega \mathbb{A}[Du, D(\Delta^-_{s,h}(\rho^2\Delta_{s,h}(u-a)))]\,\mathrm{d}x = 0$$

and, rewriting this equation,

$$\int_\Omega \mathbb{A}[\Delta^+_{s,h}Du, \rho^2 D\Delta_{s,h}(u-a) + 2\rho D\rho \otimes \Delta_{s,h}(u-a)]\,\mathrm{d}x = 0.$$

From here we deduce the *reverse Poincaré inequality with increasing supports* (colloquially termed a *Caccioppoli* inequality)

$$\int_{\mathrm{B}(x_0,r)} |\Delta_{s,h}Du|^2\,\mathrm{d}x \leqslant \frac{c}{(R-r)^2}\int_{\mathrm{B}(x_0,R)} |Du|^2\,\mathrm{d}x. \tag{1.7}$$

In consequence, $u \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. Iterating, we then find that $\partial^\alpha u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and thus $u \in \bigcap_{k\in\mathbb{N}} \mathrm{W}^{k,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ so that $u \in \mathrm{C}^\infty(\Omega;\mathbb{R}^N)$.

- L^p*-scale.* Suppose we are concerned with minima of the variational problem

$$\text{to minimise } \mathscr{F}[v] := \int_\Omega \tfrac{1}{2}|Dv|^2 - \langle v, f\rangle\,\mathrm{d}x$$

over $\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^N)$. Minima – if existent – satisfy the Euler-Lagrange equation

$$-\Delta v = f.$$

Bulding on the fundamental solution representation, we can write

$$v(x) = c_n \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\,\mathrm{d}y, \qquad x \in \mathbb{R}^n.$$

For the gradients, this implies the bound

$$|Dv(x)| \leqslant c_n \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}}\,\mathrm{d}y.$$

At this stage, we introduce the *Riesz potential operator*

$$\mathcal{I}_\alpha(f)(x) := \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}}\,\mathrm{d}y.$$

As we will see as the text evolves, $\mathcal{I}_1\colon \mathrm{L}^p(\mathbb{R}^n) \to \mathrm{L}^{\frac{np}{n-p}}(\mathbb{R}^n)$ provided $1 < p < n$. Thus, if $f \in \mathrm{L}^p(\Omega)$, then $Dv \in \mathrm{L}^{\frac{np}{n-p}}(\Omega;\mathbb{R}^n)$ – which is a regularity improvement due to $\frac{np}{n-p} > p$. A related question is given by equations where $f = \mathrm{div}(g)$ and so $\Delta u = f$ reduces to

$$-\mathrm{div}(\nabla u) = \mathrm{div}(g),$$

and we aim to transfer the regularity properties of g to ∇u. Heuristically motivated by 'cancelling' the divergence, we would expect that ∇u should share the same regularity as g – but this is not necessarily true. The underlying reason for this is the following formal computation:

$$|Dv(x)| = c_n \left| \int_{\mathbb{R}^n} D_x \frac{\operatorname{div}(g)(y)}{|x-y|^{n-2}}\,\mathrm{d}y \right| \leqslant c \int_{\mathbb{R}^n} \frac{|g(y)|}{|x-y|^n}\,\mathrm{d}y.$$

In this setting, the integral kernel on the very right-hand side has a *non-integrable* singularity – of order n – and thus needs to be interpreted by the Cauchy principal value. The corresponding operator then gives rise to a singular integral, which in the present situation maps $\mathrm{L}^p(\mathbb{R}^n) \to \mathrm{L}^p(\mathbb{R}^n)$ for if $1 < p < \infty$. However, if $p \in \{1, \infty\}$, this *is not the case in general.* The transfer of regularity from g to Dv is one of the key aspects of *potential theory.*

The situation becomes more intricate when considering the non-linear situation, where the *linear Laplacean operator* is replaced by non-linear differential expressions. Heuristically, we then have two options: Either a (nonlinear) differential operator $A[D] = \operatorname{div}(\mathbb{A}(D))$, for which we aim to set up a similar potential theory, admits *linearisation.* By this we understand to compare the system at our disposal with linear systems, for which results as surveyed above are available, and to thereby transfer regularity properties from g to the natural quantity $\mathbb{A}[D]$.

1.3. **Two perspectives on regularity.** Regularity is often visible on the level of minimising sequences. We re-embark on the variational principle (1.2) and firstly ask whether – as a very basic improvement – we can boost the weak convergence of certain minimising sequences to strong convergence. To this end, we recall the following theorem due to Riesz and Kolmogorov:

Lemma 1.10 (Riesz-Kolmogorov)**.** *Let $1 \leqslant p < \infty$. A subset $M \subset \mathrm{L}^p(\mathbb{R}^n)$ is relatively compact in $\mathrm{L}^p(\mathbb{R}^n)$ if and only if the following hold:*

(a) *M is bounded in $\mathrm{L}^p(\mathbb{R}^n)$,*
(b) *$\sup_{f\in M} \int_{\mathbb{R}^n} |f(x+h) - f(x)|^p\,\mathrm{d}x \to 0$ as $|h| \to 0$,*
(c) *$\sup_{f\in M} \int_{\mathbb{R}^n\setminus \mathrm{B}(0,R)} |f(x)|^p\,\mathrm{d}x \to 0$, $r \nearrow \infty$.*

The informal output regarding *failure of strong convergence* in L^p thus is oscillation and concentration. This can be equally manifested by the Vitali convergence theorem. Condition (b) of the preceding theorem resembles the equicontinuity of the Ascoli-Arzelá theorem from elementary functional analysis, which we recall here for compact underlying sets:

Lemma 1.11 (Ascoli-Arzelá)**.** *Let (X, d) be a compact metric space and let $\mathcal{F} \subset \mathrm{C}(X;\mathbb{R})$. Then the following are equivalent:*

- *$\mathcal{F}$ is relatively compact in $\mathrm{C}(X;\mathbb{R})$.*
- *$\mathcal{F}$ is pointwisely bounded (i.e., for each $x \in X$ there exists $c_x > 0$ such that $|f(x)| \leqslant c_x$ for all $f \in \mathcal{F}$) and equicontinuous (i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $d(x,y) < \delta$ there holds $\sup_{f\in\mathcal{F}} |f(x) - f(y)| < \varepsilon$).*

As an upshot, compactness is 'essentially' equivalent to equicontinuity. A measure for the degree of equicontinuity is provided by *moduli of continuity.* Since we are in the function space situation, such moduli of continuity must respect the space scale we are working with. The metaprinciple thus is

$$\boxed{\text{Regularity often stems from quantitative compactness results.}}$$

This can be seen easiest at the Caccioppoli inequality, which yields a local compactness result for the gradients – and in turn improves the integrability.

April 1, 2019

2. Higher Sobolev Regularity for Variational Problems

2.1. **Motivation.** The present chapter aims to give an introduction to *higher Sobolev regularity results.* Beyond being of independent interest as a regularity result per se, this section serves to give explicit motivation why we strive for higher Sobolev regularity in a variety of contexts.

(a). *Function space moduli of continuity.* We recall from the preceding section that if $\mathbb{A} \in \mathbb{S}(\mathbb{R}^{N\times n})$ is an elliptic bilinear form on $\mathbb{R}^{N\times n}$, then $\mathbb{A}$-weakly harmonic maps $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$ are actually of class C^∞. As we discussed above, the *most basic regularity* assertion is boost from weak to strong convergence of minimising sequences in $\mathrm{L}^p_{(\mathrm{loc})}$. By the Riesz-Kolmogorov criterion, the requisite compactness is essentially equivalent to having a function space modulus of continuity for the entire minimising sequences. This led us to the metaprinciple '*Regularity often stems from quantitative compactness results*', see above.

This approach, originally due to Shiffman, crucially hinges on the difference quotient characterisation of Sobolev spaces. We recall the statement in some higher generality:

Lemma 2.1. *Let $1 < p < \infty$ and let $u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$. Then we have $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$ if and only if for any $\omega \Subset \mathbb{R}^n$ there holds*

$$\sup_{|h|\leqslant 1}\sum_{s=1}^{n}\|\Delta_{s,h}u\|_{\mathrm{L}^p(\omega;\mathbb{R}^N)} < \infty. \tag{2.1}$$

Note that this lemma is sharp in the sense that it *does not extend* to $p = 1$, a fact that shall become visible from the proof. Indeed, $p = 1$ characterises the space $\mathrm{BV}_{\mathrm{loc}}$ to be discussed below.

Proof of Lemma 2.1. Since the statement is local, it is no loss of generality to assume that $u \in \mathrm{W}^{1,p}_c(\mathbb{R}^n;\mathbb{R}^N)$ first; in particular, u is a compactly supported Sobolev map. By smooth approximation in the norm topology of $\mathrm{W}^{1,p}(\mathbb{R}^n;\mathbb{R}^N)$, we find $(u_j) \subset \mathrm{C}^\infty_c(\mathbb{R}^n;\mathbb{R}^N)$ such that $u_j \to u$ in $\mathrm{W}^{1,p}(\mathbb{R}^n;\mathbb{R}^N)$ as $j \to \infty$. The fundamental theorem of calculus in conjunction with the chain rule yields for all $\omega \Subset \mathbb{R}^n$, $h > 0$ and $s \in \{1, ..., n\}$

$$\begin{aligned}\int_\omega \left|\frac{u_j(x+he_s)-u_j(x)}{h}\right|^p \mathrm{d}x &= \int_\omega \left|\int_0^1 Du_j(x+the_s)\cdot e_s \,\mathrm{d}t\right|^p \mathrm{d}x \\ &\leqslant \int_0^1\int_\omega |Du_j(x+the_s)|^p \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \int_{\omega+\mathrm{B}(0,2)} |Du_j|^p \,\mathrm{d}x\end{aligned}$$

for if h is sufficiently small. Passing to the limit $j \to \infty$, we obtain

$$\int_\omega \left|\frac{u(x+he_s)-u(x)}{h}\right|^p \mathrm{d}x \leqslant \int_{\omega+\mathrm{B}(0,2)} |Du|^p \,\mathrm{d}x,$$

and so (2.1) holds. For the reverse direction, let $u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$ satisfy (2.1). We need to establish that u possesses weak derivatives of order one, all of which belong to $\mathrm{L}^p_{\mathrm{loc}}$ themselves. Let $h>0$ and $i\in\{1,...,n\}$ be arbitrary, and suppose that $\varphi\in\mathrm{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^N)$ is a given test map. We then observe that – with the usual definition of forward and backward difference quotients $\Delta_{i,\pm h}$ –

$$\int_{\mathbb{R}^n} u\cdot\Delta_{i,h}\varphi\,\mathrm{d}x = -\int_{\mathbb{R}^n}(\Delta_{i,-h}u)\cdot\varphi\,\mathrm{d}x. \tag{2.2}$$

Due to assumption (2.1), we have

$$\sup_{0\leqslant h\leqslant 1}\|\Delta_{i,-h}u\|_{\mathrm{L}^p(\mathrm{spt}(\varphi);\mathbb{R}^N)}<\infty.$$

By the Banach-Alaoglu-Bourbaki theorem, $1<p<\infty$ consequently implies that there exists a sequence $(h_j)\subset\mathbb{R}_{\geqslant 0}$ with $h_j\searrow 0$ and some $u_i\in\mathrm{L}^p(\mathrm{spt}(\varphi);\mathbb{R}^N)$ such that $\Delta_{i,-h_j}u\rightharpoonup u_i$ weakly in $\mathrm{L}^p(\mathrm{spt}(\varphi);\mathbb{R}^N)$ as $j\to\infty$. In consequence, we find

$$\int_{\mathbb{R}^n} u\cdot\Delta_{i,h_j}\varphi\,\mathrm{d}x \stackrel{(2.2)}{=} -\int_{\mathbb{R}^n}(\Delta_{i,-h_j}u)\cdot\varphi\,\mathrm{d}x \stackrel{j\to\infty}{\longrightarrow} -\int_{\mathbb{R}^n}u_i\cdot\varphi\,\mathrm{d}x,$$

and the dominated convergence theorem readily yields that the very left-hand side of the previous equation converges to $\int_{\mathbb{R}^n}u\cdot\partial_i\varphi\,\mathrm{d}x$. In consequence, by arbitrariness of φ and $i\in\{1,...,n\}$, u has weak derivatives of order one and since $\partial_i u=u_i\in\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$, the claim is fully established. □

Working from here, we see that the Caccioppoli-type inequality (1.7) yields a quantitative modulus of continuity

$$\Big(\int_{\mathrm{B}(x_0,r)}|\tau_{s,h}Du_j|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\leqslant C|h|\Big(\int_{\mathrm{B}(x_0,R)}\frac{|Du_j|^2}{(R-r)^2}\,\mathrm{d}x\Big)^{\frac{1}{2}}\leqslant C(r,R)|h| \tag{2.3}$$

whenever (u_j) is a minimising sequence of the associated variational integral

$$v\mapsto\int_\Omega\mathbb{A}[\nabla v,\nabla v]\,\mathrm{d}x$$

to some fixed Dirichlet boundary data, say. This modulus of continuity is quite strong; note that, with suitable interpretation, the above estimate (2.3) amounts to an L^2-Lipschitz condition.

At this stage, we take the opportunity to motivate the introduction of the function space machinery as shall be addressed below in Section 2.2. Similar to weakly $\mathbb{A}$-harmonic maps (where the bilinear form $\mathbb{A}$ is fixed), we may consider the situation where $\mathbb{A}(x)$ depends on the spatial variable x. More precisely, let $\mathbb{A}\colon\Omega\to\mathbb{S}(\mathbb{R}^{N\times n})$ be a map such that there holds

$$\lambda|\xi|^2\leqslant\mathbb{A}(x)[\xi,\xi]\leqslant\Lambda|\xi|^2\qquad\text{for all }x\in\mathbb{R}^n,\ \xi\in\mathbb{R}^{N\times n}.$$

When $\mathbb{A}\colon\Omega\to\mathbb{S}(\mathbb{R}^{N\times n})$ is not assumed differentiable but merely Hölder continuous, the difference quotient outlined previously is bound to fail: Indeed, going through the considerations that led to (1.7), we see that when we perform the integration by parts for the difference quotients, one difference quotient neccessarily applies to $\mathbb{A}(-)$. The arising terms cannot be suitably controlled, and thus the estimation becomes useless.

A key observation due to MINGIONE was to realise that an adaptation of the difference quotient method still can be employed *provided we leave the realm of full* *difference quotients and pass to fractional space scales.* In this way, the corresponding metaprinciple reads as:

Hölder continuity of the coefficients translates to suitable fractional differentiability of minima/weak solutions.

Again, we shall obtain a quantitative modulus of continuity for the gradients of minimising sequences, now with a power $|h|^s$ instead of $|h| = |h|^1$ as in (2.3). Here, s will depend on the Hölder continuity of $\mathbb{A}$ with respect to x. Essentially, the approach we follow give here is contained in

- Mingione, G.: The singular set of solutions to non-differentiable elliptic systems. Arch. Ration. Mech. Anal. 166, 287–301 (2003).

(b). *(p, q)-type growth behaviour and representations of relaxed functionals.* By violation of the p-growth hypothesis (p-growth), a variety of mathematical models or variational problems do *not* quite fit into the framework of the variational principles discussed in Section 1. Some times we encounter integrands $F\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ which merely satisfy

$$c_1|z|^p - c_2 \leqslant F(x,z) \leqslant c_3(1+|z|^q) \qquad \text{for all } x \in \Omega,\ z \in \mathbb{R}^{N\times n}, \tag{2.4}$$

where now $1 < p < q < \infty$. Running the direct method for a suitable Dirichlet principle as outlined in the introduction, we see that the

- the associated variational integral $\mathscr{F}$ is defined on $\mathrm{W}^{1,q}$ by the upper bound in (2.4),
- but coerciveness is only provided in $\mathrm{W}^{1,p}$ by the lower bound in (2.4).

Compactness thus forces us to work on the larger space $\mathrm{W}^{1,p}$. Hence we must firstly extend $\mathscr{F}$ from $\mathrm{W}^{1,q}(\Omega;\mathbb{R}^N)$ to $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$. This is usually accomplished by means of the Lebesgue-Serrin-type extension

$$\overline{\mathscr{F}}[u] := \inf\left\{\liminf_{j\to\infty}\mathscr{F}[u_j]\colon \begin{array}{c}(u_j)\subset \mathrm{W}^{1,q}(\Omega;\mathbb{R}^N),\\ u_j \rightharpoonup u \text{ in } \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)\end{array}\right\}, \qquad u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N).$$

The approach that underlies this definition is *to extend to a larger space by lower semicontinuity.* As a drawback, things become more abstract, and one may wonder whether *we can represent the functional $\overline{\mathscr{F}}$ as a variational integral:*

$$\overline{\mathscr{F}}[u] = \int_\Omega G(Du)\,\mathrm{d}x,$$

at least for maps $u \in \mathrm{W}^{1,p}_c(\Omega;\mathbb{R}^N)$, say, where G is a suitable integrand strictly related to F. Here, the core topic of this chapter steps in, namely, to give criteria on

- the regularity and ellipticity of F
- together with the relation between p and q,

such that (at least) minimisers of the functional $\mathscr{F}$ belong to $\mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. *This is another face of higher Sobolev regularity*, to be achieved by a similar procedure as is done for (a) from above. We will see that, subject to and additional, suitable $\mathrm{C}^{0,\alpha}$-Hölder continuity of F with respect to its first variable, the relevant bound reads as

$$\frac{q}{p} < \frac{n+\alpha}{n}.$$

Note that if F is of class C^2 with respect to the joint variable (x,z), then the corresponding bound improves to

$$\frac{q}{p} < 1 + \frac{2}{n}.$$

We shall discuss this in much more detail as the chapter evolves and supply counterexamples for *irregularity* provided these conditions are *not* met by the integrands. To conclude with, it is interesting to note that if $n \to \infty$, this bound enforces $q \searrow p$, showing the particular dimensional impact on regularity. Indeed, such results are encountered frequently: The higher the dimension n, the worse the regularity of minima.

2.2. **A quick introduction to Besov spaces.** To measure the fractional smoothness of minima, we wish to come up with a desirably flexible scale of function spaces. A priori, it is far from clear how to come up with a function space scale that is well-suited for measuring *fractional differentiability* or smoothness, respectively. To outline the basic underlying ideas, we firstly present a general approach to fractional smoothness. For the purposes of this chapter, this initial definition – though far reaching – is not very well adapted. Hence we shall discuss another, related way that eventually will turn out equivalent for a vast range of parameters.

To begin with, let us recall the so-called *Bessel potential spaces* as known from the usual PDE course syllabus. Within the L^2-framework, one establishes by use of smooth approximation and the Plancherel theorem that $u\colon \mathbb{R}^n \to \mathbb{R}$ belongs to $\mathrm{W}^{k,2}(\mathbb{R}^n)$, $k \in \mathbb{N}$, if and only if

$$\int_{\mathbb{R}^n} \Big|(1+|\xi|^2)^{\frac{k}{2}} |\mathscr{F}u(\xi)|\Big|^2 \,\mathrm{d}\xi < \infty. \tag{2.5}$$

One then realises that we might also take k to be non-integer here, and this leads us to the Bessel potential spaces to be defined as

$$\mathrm{H}^{s,p}(\mathbb{R}^n) := \Big\{u \in \mathrm{L}^p(\mathbb{R}^n)\colon\ (\xi \mapsto (\sqrt{1+|\xi|}^{\,s}\,\widehat{u}(\xi)) \in \mathrm{L}^p(\mathbb{R}^n)\Big\}.$$

Let us have a closer look at (2.5): When passing to the Fourier transform, ξ takes the role of the *frequency variable*, the latter being reflected by the functions $h_\xi\colon x \mapsto \exp(-\mathrm{i}\langle x,\xi\rangle) = \cos(\langle x,\xi\rangle) - \mathrm{i}\sin(\langle x,\xi\rangle)$ that appear in the definition of the Fourier transform. In consequence, larger values of $|\xi|$ correspond to higher frequencies and thus stronger oscillatory behaviour of the functions h_ξ. By the very structure of $\mathscr{F}^{-1}$, we have (at least for $u \in \mathscr{S}(\mathbb{R}^n)$)

$$u(x) = \mathscr{F}^{-1}[\mathscr{F}u](x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} [\mathscr{F}u(\xi)] \exp(\mathrm{i}\langle x,\xi\rangle)\,\mathrm{d}\xi.$$

From this identity we see that u itself is built up from functions with different frequencies h_ξ, each being weighted by $\mathscr{F}u(\xi)$. Here our intuitive understanding is that *the less oscillatory a function is, the more regular regular it is.* From a heuristic perspective we thus obtain the following conclusion: The *smaller the Fourier transform $\mathscr{F}u(\xi)$ is for larger values of ξ – or, put differently, the faster $\mathscr{F}u(\xi)$ decays as $|\xi|$ increases – the less high frequencies contribute to u itself* and so the more regular u will be.

It is now a matter of how this decay of the Fourier transform is measured. To explain the procedure in detail, we decompose the *phase space* $\mathbb{R}^n$ into overlapping dyadic annuli A_j ($j \in \mathbb{N}$) which are defined by

$$A_j := \{\xi \in \mathbb{R}^n :\ 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}. \tag{2.6}$$

In a next step, we choose a partition of unity (φ_j) subject to (A_j) and write

$$\mathscr{F}u(\xi) = \Big(\sum_{j=0}^{\infty} \varphi_j(\xi)\Big)\mathscr{F}u(\xi) = \sum_{j=0}^{\infty} \big(\varphi_j(\xi)\mathscr{F}u(\xi)\big).$$

Transforming back, we obtain

$$u(x) = \sum_{j=0}^{\infty} \mathscr{F}^{-1}_{\xi\mapsto x}\big[\varphi_j(\xi)\mathscr{F}u(\xi)\big]$$

and each of the *building blocks* $u_j := \mathscr{F}^{-1}_{\xi\mapsto x}\big[\varphi_j(\xi)\mathscr{F}u(\xi)\big]$ is the inverse Fourier transform of $\mathscr{F}u(\xi)$ restricted to the frequency range $2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}$. We now consider the L^p-norms $\|u_j\|_{\mathrm{L}^p(\mathbb{R}^n)}$, thus providing us with a sequence $(2^{js}\|u_j\|_{\mathrm{L}^p(\mathbb{R}^n)})_{j=0}^{\infty}$. To capture the regularity of u via the speed of decay, we take advantage of the sequence spaces $\ell_q(\mathbb{N}_0)$ and examine

to which $\ell_q(\mathbb{N}_0)$ the sequence $(2^{js}\|u_j\|_{\mathrm{L}^p(\mathbb{R}^n)})_{j=0}^{\infty}$ belongs to.

If $(2^{js}\|u_j\|_{\mathrm{L}^p(\mathbb{R}^n)})_{j=0}^{\infty} \in \ell_q(\mathbb{N}_0)$, we shall say that u belongs to the Besov space $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$.

Motivated by these heuristics, we now turn to the rigorous setup and begin with constructing the dyadic partition of unity of phase space. Let $\varphi_0 \in \mathrm{C}^\infty_c(\mathbb{R}^n)$ be such that

$$\varphi_0(\xi) = 1 \text{ for } |\xi| \leqslant 1 \quad \text{and} \quad \varphi_0(\xi) = 0 \text{ for } |\xi| > 2. \tag{2.7}$$

To facilitate exposition, we shall moreover assume that $\mathrm{dist}(\mathrm{spt}(\varphi_0); \partial\mathrm{B}(0,2)) > 0$. We put, for $j \in \mathbb{N}$,

$$\varphi_j(\xi) := \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi), \qquad \xi \in \mathbb{R}^n. \tag{2.8}$$

Then the following hold:

(a) $\mathrm{spt}(\varphi_0) \subset \overline{\mathrm{B}}(0,2)$. This is clear by definition of φ_0.

(b) $\mathrm{spt}(\varphi_j) \subset \{\xi \in \mathbb{R}^n :\ 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}$ for all $j \in \mathbb{N}$. Indeed, if $|\xi| > 2^{j+1}$, then $2^{-j}|\xi| > 2$ and so $2^{-j+1}|\xi| > 4$. Therefore $\varphi_j(\xi) = 0-0 = 0$ in this case. If, $|\xi| < 2^{j-1}$, then $2^{-j}|\xi| < \frac{1}{2}$ and so $2^{-j+1}|\xi| < 1$. Thus, $\varphi_j(\xi) = 1-1 = 0$ in this case, and the claim follows.

(c) If $j,k \in \mathbb{N}$ satisfy $|j-k| \geqslant 2$, then $\mathrm{spt}(\varphi_j) \cap \mathrm{spt}(\varphi_k) = \emptyset$. To see this, let without loss of generality $k > j$ and $k - j \geqslant 2$. Then $2^{k-j} \geqslant 2^2$ and thus $2^{k-1} \geqslant 2^{j+1}$. In consequence, the claim follows in view of (b).

(d) Lastly, the sequence $(\varphi_j)_{j=0}^{\infty}$ forms a partition of unity:

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n. \tag{2.9}$$

In fact, let $\xi \in \mathbb{R}^n$ and pick $k \in \mathbb{N}$ such that $|\xi| \leqslant 2^k$. Then, if $j \geqslant k+1$, $\varphi_j(\xi) = 0$ and so, by a telescope sum argument in the ultimate step,

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = \sum_{j=0}^{k} \varphi_j(\xi) = \varphi_0(\xi) + \sum_{j=1}^{k} \big(\varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi)\big) = \varphi_0(2^{-k}\xi),$$

and since $|\xi| \leqslant 2^k$, $2^{-k}|\xi| \leqslant 1$ and so $\varphi_0(2^{-k}\xi) = 1$.

Now let $f \in \mathscr{S}(\mathbb{R}^n)$ (so that $\widehat{f} \in \mathscr{S}(\mathbb{R}^n)$ as well) and write for $\xi \in \mathbb{R}^n$

$$\widehat{f}(\xi) = \underbrace{\Big(\sum_{j=0}^{\infty}\varphi_j(\xi)\Big)}_{=1}\widehat{f}(\xi) = \sum_{j=0}^{\infty}(\varphi_j(\xi)\widehat{f}(\xi)) =: \sum_{j=0}^{\infty} f_j(\xi), \tag{2.10}$$

where we have put $f_j := \varphi_j\widehat{f}$. Moreover, we observe that for all $x \in \mathbb{R}^n$ there holds

$$f(x) = \mathscr{F}^{-1}_{\xi\mapsto x}\big[\sum_{j=0}^{\infty}\varphi_j(\xi)\hat{f}(\xi)\big] = \sum_{j=0}^{\infty}\mathscr{F}^{-1}_{\xi\mapsto x}(\varphi_j(\xi)\hat{f}(\xi)) = \sum_{j=0}^{\infty}(\check{\varphi}_j * f)(x).$$

However, in order to introduce smoothness spaces, it is advisable to admit more general elements than functions f belonging to the Schwartz class. Instead, we employ the tempered distributions $\mathscr{S}'(\mathbb{R}^n)$. Then, taking advantage of the multiplication theorems for tempered distributions, we can write

$$\widehat{f} = \sum_{j=0}^{\infty}(\varphi_j\widehat{f}) \quad \text{and} \quad f = \sum_{j=0}^{\infty}\mathscr{F}^{-1}(\varphi_j\widehat{f}) =: \sum_{j=0}^{\infty}\varphi_j(D)f. \tag{2.11}$$

for $f \in \mathscr{S}'(\mathbb{R}^n)$. Note carefully that the respective operations have to be interpreted accordingly. Now we are ready to formalise the above ideas and make the following

Definition 2.2 (Besov Spaces)**.** *Let $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$. The* Besov space $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$ *precisely consists of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that*

$$\|f\|^q_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)} := \sum_{j=0}^{\infty} 2^{jsq}\|\varphi_j(D_x)f\|^q_{\mathrm{L}^p(\mathbb{R}^n)} < \infty \qquad \text{if } 1 \leqslant q < \infty, \tag{2.12}$$

$$\|f\|^q_{\mathrm{B}^s_{p,\infty}(\mathbb{R}^n)} := \sup_{j\in\mathbb{N}_0} 2^{js}\|\varphi_j(D_x)f\|_{\mathrm{L}^p(\mathbb{R}^n)} < \infty \qquad \text{if } q = \infty. \tag{2.13}$$

By a result due to Peetre, the particular choice of the partition of unity of the phase space does *not* have a special standing and essentially others will do as well, giving rise to the same definition of Besov spaces.

2.2.1. *Sequence spaces $\ell^s_q(\mathbb{Z}; X)$.* The purpose of this section is to present a more convenient formulation of the criteria for a function to belong to some space $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$. More precisely, given $1 \leqslant q \leqslant \infty$, $s > 0$, a subset[2] $A \subset \mathbb{Z}$ and a normed space $(X, \|\cdot\|_X)$, we say that an X–valued sequence $(a_j)_{j\in A}$ belongs to $\ell^s_q(A; X)$ if and only if

$$\begin{aligned} &\|(a_j)_{j\in A}\|^q_{\ell^s_q(A;X)} := \sum_{j\in A} 2^{jsq}\|a_j\|^q_X < \infty, \qquad 1 \leqslant q < \infty \\ &\|(a_j)_{j\in A}\|_{\ell^s_\infty(A;X)} := \sup_{j\in A} 2^{js}\|a_j\|_X < \infty, \qquad q = \infty. \end{aligned} \tag{2.14}$$

With these conventions, we have the following characterisation of a tempered distribution $f \in \mathscr{S}'(\mathbb{R}^n)$:

$$f \in \mathrm{B}^s_{p,q}(\mathbb{R}^n) \ \text{ if and only if } \ (\varphi_j(D_x)f) \in \ell^s_q(\mathbb{N}_0; \mathrm{L}^p(\mathbb{R}^n)) \tag{2.15}$$

[2]For the present situation, $A = \mathbb{N}_0$ will do. Since no difficulties arise when dealing with more general case of $A \subset \mathbb{Z}$ and because we need this generalisation later for homogeneous Besov spaces, we directly work on sequences indexed by $A \subset \mathbb{Z}$.

In order to obtain more insight into the Besov spaces as introduced in Definition 2.2, we first investigate the Banach space-valued sequence spaces $\ell_q^s(\mathbb{N}_0; X)$ in more detail.

Lemma 2.3. *Let $1 \leqslant q \leqslant \infty$, $s \in \mathbb{R}$ and $A \subset \mathbb{Z}$. Then $(\ell_q^s(A; X), \|\cdot\|_{\ell_q^s(A;X)})$ is Banach.*

Proof. The proof is entirely standard and exactly follows the arguments known for the usual sequence spaces $\ell^p(\mathbb{N})$. ☐

Because of (2.15), many results such as ad–hoc embedding results of Besov spaces can be reduced to those of sequence spaces. Omitting the proof, we record

Proposition 2.4. *Let $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$ and $s \in \mathbb{R}$. Then the following hold:*

(a) $(\mathrm{B}_{p,q}^s(\mathbb{R}^n), \|\cdot\|_{\mathrm{B}_{p,q}^s(\mathbb{R}^n)})$ *is a Banach space.*
(b) $\mathscr{S}(\mathbb{R}^n) \hookrightarrow \mathrm{B}_{p,q}^s(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n)$.
(c) $\mathscr{S}(\mathbb{R}^n)$ *is dense in* $\mathrm{B}_{p,q}^s(\mathbb{R}^n)$.

Proof. The proof of this proposition shall be given in the lecture notes and is not examinable. ☐

To get an idea of the interplay between the single parameters in the definition of the spaces $\mathrm{B}_{p,q}^s$, we shall prove an important embedding result in Proposition 2.6. Beforehand, we remind the reader of the following

Lemma 2.5. *Let $1 \leqslant p \leqslant q \leqslant \infty$, $s > 0$, $A \subset \mathbb{Z}$ and let $(X, \|\cdot\|)$ be a normed space. Then the embedding $\ell^p(A; X) \subset \ell^q(A; X)$ is continuous.*

Proof. The proof is entirely standard and directly follows from the arguments known for the usual sequence spaces $\ell^p(\mathbb{N})$. ☐

Proposition 2.6. *Let $s > 0$, $1 \leqslant p, q \leqslant \infty$. Then*

$$\mathrm{B}_{\infty}^{s,p}(\mathbb{R}^n) \hookrightarrow \mathrm{B}_1^{s-\varepsilon,p}(\mathbb{R}^n) \hookrightarrow \mathrm{B}_p^{s-\varepsilon,p}(\mathbb{R}^n) \qquad \text{for all } 0 < \varepsilon < s. \tag{2.16}$$

Moreover, if $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, then

$$\mathrm{B}_{p,q_1}^s(\mathbb{R}^n) \subset \mathrm{B}_{p,q_2}^s(\mathbb{R}^n). \tag{2.17}$$

Proof. Ad (2.16). Let $u \in \mathrm{B}_{\infty}^{s,p}(\mathbb{R}^n)$. Then, by the convergence properties of the geometric series,

$$\begin{aligned} \|u\|_{\mathrm{B}_1^{s-\varepsilon,p}(\mathbb{R}^n)} = \sum_{j=0}^{\infty} 2^{j(s-\varepsilon)} \|\varphi_j(D_x)f\|_{\mathrm{L}^p(\mathbb{R}^n)} &\leqslant \big(\sup_{j\in\mathbb{N}_0} 2^{js} \|\varphi_j(D_x)f\|_{\mathrm{L}^p(\mathbb{R}^n)}\big) \times \sum_{j=0}^{\infty} 2^{-j\varepsilon} \\ &= \frac{1}{1+2^{-\varepsilon}} \|u\|_{\mathrm{B}_{\infty}^{s,p}(\mathbb{R}^n)}. \end{aligned}$$

This proves the first embedding, whereas the second is an immediate consequence of Lemma 2.5 by virtue of (2.15). Ad (2.17). This is a direct consequence of Lemma 2.5. The proof is complete. ☐

The previous proposition gives rise to the following metaprinciple:

- If s and p are fixed, then $\mathrm{B}_{p,q}^s(\mathbb{R}^n)$ becomes larger provided q increases. As such, $\mathrm{B}_{p,1}^s(\mathbb{R}^n)$ is the smallest space and $\mathrm{B}_{p,\infty}^s(\mathbb{R}^n)$ the largest.

- If p is fixed, we may sacrifice some smoothness (s) to *decrease* the fine parameter q.

The distinction offered by the parameter q motivates the terminology of *fine parameter q*. The embeddings of Proposition 2.6 can be iterated and so further inclusions can be obtained. Note, however, that each of the inclusion is strict in general. On the other hand, if we examplarily want to switch from $\mathrm{B}_q^{s,p}(\mathbb{R}^n)$ to some $\mathrm{B}_{q_1}^{s_1,p_1}$ with $s_1 < s$, then this again runs under the name of *Sobolev embeddings* and will be addressed later on.

2.2.2. *The difference quotient approach.* The relevant modification of the proof of higher Sobolev regularity for systems with coefficients of class $\mathrm{C}^{0,\alpha}$, say, is based on *finite differences*. It is thus advisable to obtain a different description of the Besov scale by use of precisely those finite differences. For a function $u\colon \mathbb{R}^n \to \mathbb{R}$, $h > 0$ and $i \in \{1,...,n\}$, we put

$$\tau_{i,h}u(x) := u(x+he_i) - u(x), \qquad x \in \mathbb{R}^n.$$

We then make the following

Definition 2.7 (Besov spaces via difference quotients)**.** *Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$ and $0 < s < 1$. We say that a mapping $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $\mathrm{B}_{p,q}^{s,\mathrm{diff}}(\mathbb{R}^n)$ if and only if $u \in \mathrm{L}^p(\mathbb{R}^n)$ and the Besov seminorm*

$$[u]_{\mathrm{B}_{p,q}^{s,\mathrm{diff}}(\mathbb{R}^n)} := \sum_{i=1}^n \Big(\int_0^\infty \frac{\|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q}{h^{1+sq}}\,\mathrm{d}h\Big)^{\frac{1}{q}} \tag{2.18}$$

is finite. If $1 \leqslant p \leqslant \infty$ and $q = \infty$, then $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $\mathrm{B}_{p,\infty}^{s,\mathrm{diff}}(\mathbb{R}^n)$ if and only if $u \in \mathrm{L}^p(\mathbb{R}^n)$ and the Besov seminorm

$$[u]_{\mathrm{B}_{p,\infty}^{s,\mathrm{diff}}(\mathbb{R}^n)} := \sum_{i=1}^n \sup_{h>0} \frac{\|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}}{h^s} \tag{2.19}$$

is finite. In any case, the full Besov norm *is given by $\|u\|_{\mathrm{B}_{p,q}^{s,\mathrm{diff}}(\mathbb{R}^n)} := \|u\|_{\mathrm{L}^p(\mathbb{R}^n)} + [u]_{\mathrm{B}_{p,q}^s(\mathbb{R}^n)}$.*

The spaces $\mathrm{B}_{p,\infty}^{s,\mathrm{diff}}(\mathbb{R}^n)$ are also called *Nikolskiĭ spaces* and then denoted $\mathcal{N}^{s,p}(\mathbb{R}^n) := \mathrm{B}_{p,\infty}^{s,\mathrm{diff}}(\mathbb{R}^n)$. We will see later that $\mathrm{B}_{p,q}^{s,\mathrm{diff}}$ *often* coincides with $\mathrm{B}_{p,q}^s$ – at least in all cases that are relevant for us.

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Remark 2.8. *If $0 < s < 1$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, then $\mathrm{B}_{p,q}^{s,\mathrm{diff}}(\mathbb{R}^n) \cong \mathrm{B}_{p,q}^s(\mathbb{R}^n)$.*

The last remark, though seemingly very minor, is at the heart of the theory of Besov spaces and runs under the name of *Littlewood-Paley results*. Despite not dealing with it in the lectures, a detailled proof shall be inserted at the end of this manuscript soon. From here it is clear that

$$\mathrm{B}_{p,q_1}^{s,\mathrm{diff}}(\mathbb{R}^n) \hookrightarrow \mathrm{B}_{p,q_2}^{s,\mathrm{diff}}(\mathbb{R}^n)$$

whenever $q_1 \leqslant q_2$, and we moreover shall now mutually use the finite difference characterisation whenever the smoothness exponents provided by Remark 2.8 are met.

To get an idea of what this space scales does, we will study elementary connections of the $\mathrm{B}_{p,q}^s$-spaces and their relation to the usual Sobolev spaces first. We begin with

Lemma 2.9. *Let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $0 < t \leqslant s < 1$. Then*

$$\mathrm{B}_{p,q}^{s,\mathrm{diff}}(\mathbb{R}^n) \hookrightarrow \mathrm{B}_{p,q}^{t,\mathrm{diff}}(\mathbb{R}^n).$$

Proof. We only provide the proof for $q < \infty$; $q = \infty$ follows almost trivially. Let $u \in \mathrm{B}_{p,q}^{s,\mathrm{diff}}(\mathbb{R}^n)$ and fix $i \in \{1, ..., n\}$. We split

$$\int_0^\infty \frac{\|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q}{h^{1+tq}}\,\mathrm{d}h = \int_0^1 \frac{\|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q}{h^{1+tq}}\,\mathrm{d}h + \int_1^\infty \frac{\|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q}{h^{1+tq}}\,\mathrm{d}h =: (*).$$

Now, using $h^{-1-tq} \leqslant h^{-1-sq}$ for $0 < h < 1$ for the first and the trivial estimate $\|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant 2\|u\|_{\mathrm{L}^p(\mathbb{R}^n)}$ for the second integral, we obtain

$$\begin{aligned}(*) &\leqslant \int_0^1 \frac{\|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q}{h^{1+sq}}\,\mathrm{d}h + 2^q\|u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q \int_1^\infty \frac{\mathrm{d}h}{h^{1+tq}} \\ &\leqslant [u]_{\mathrm{B}_{p,q}^{s,\mathrm{diff}}(\mathbb{R}^n)}^q + \frac{2^q}{tq}\|u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q.\end{aligned}$$

Summing over all $i \in \{1, ..., n\}$ we then easily obtain the desired conclusion. □

Next we turn to the relation of $\mathrm{B}_{p,q}^s(\mathbb{R}^n)$ to $\mathrm{W}^{1,p}(\mathbb{R}^n)$:

Theorem 2.10 ($\mathrm{W}^{1,p}$ in the Besov scale)**.** *For any $0 < s < 1$, $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$ there holds*

$$\mathrm{W}^{1,p}(\mathbb{R}^n) \hookrightarrow \mathrm{B}_{p,q}^{s,\mathrm{diff}}(\mathbb{R}^n),$$

the embedding norm only depending on n, q, s.

Proof. Let $u \in \mathrm{W}^{1,p}(\mathbb{R}^n)$. We invoke $\|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant |h|\|\,\mathrm{D}u\|_{\mathrm{L}^p(\mathbb{R}^n)}$ to estimate

$$\begin{aligned}[u]_{\mathrm{B}_{p,q}^{s,\mathrm{diff}}(\mathbb{R}^n)} &= \sum_{i=1}^n \Big(\int_0^\infty \frac{\|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q}{h^{1+sq}}\,\mathrm{d}h\Big)^{\frac{1}{q}} \\ &= \sum_{i=1}^n \Big(\int_0^1 \frac{\|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q}{h^{1+sq}}\,\mathrm{d}h + \int_1^\infty \frac{\|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q}{h^{1+sq}}\,\mathrm{d}h\Big)^{\frac{1}{q}} \\ &\leqslant \sum_{i=1}^n \Big(\|\partial_i u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q \int_0^\infty \frac{\mathrm{d}h}{h^{1+(s-1)q}} + 2^q\|u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q \int_1^\infty \frac{\mathrm{d}h}{h^{1+sq}}\Big)^{\frac{1}{q}} \\ &\leqslant \sum_{i=1}^n \|\partial_i u\|_{\mathrm{L}^p(\mathbb{R}^n)} \Big(\frac{1}{(1-s)q}\Big)^{\frac{1}{q}} + 2 \\ &= \frac{n}{(1-s)q}\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)} + \frac{2^q}{sq}\|u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q \leqslant C\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)},\end{aligned}$$

and since trivially $u \in \mathrm{L}^p(\mathbb{R}^n)$, we obtain the estimate $\|u\|_{\mathrm{B}_{p,q}^s(\mathbb{R}^n)} \leqslant C\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)}$ and hence the first embedding. □

Remark 2.11 (Renormalised limit passage)**.** *Note that the above proof shows that*

$$(s(1-s)q)^{\frac{1}{q}}\Big(\int_0^\infty \|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q \frac{\mathrm{d}h}{h^{1+sq}}\Big)^{\frac{1}{q}} \leqslant s^{\frac{1}{q}}\|\partial_i u\|_{\mathrm{L}^p(\mathbb{R}^n)} + 2(1-s)^{\frac{1}{q}}\|u\|_{\mathrm{L}^p(\mathbb{R}^n)}.$$

This suggests the brave guess that we have for $u \in \mathrm{L}_{\mathrm{loc}}^1(\mathbb{R}^n)$ there holds

$$u \in \mathrm{L}^p(\mathbb{R}^n) \quad \textit{if and only if} \quad \liminf_{s \searrow 0}(s(1-s)q)^{\frac{1}{q}}\Big(\int_0^\infty \|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q \frac{\mathrm{d}h}{h^{1+sq}}\Big)^{\frac{1}{q}} < \infty$$

and, for $u \in \mathrm{L}^p(\mathbb{R}^n)$,

$$u \in \mathrm{W}^{1,p}(\mathbb{R}^n) \quad \textit{if and only if} \quad \liminf_{s\nearrow 1}(s(1-s)q)^{\frac{1}{q}}\Big(\int_0^\infty \|\tau_{i,h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q \frac{\mathrm{d}h}{h^{1+sq}}\Big)^{\frac{1}{q}} < \infty.$$

These limit passages indeed hold, and will be inserted here at some other point. The corresponding renormalisation factors $(s(1-s)q)^{\frac{1}{q}}$ *on the relevant left-hand sides are all-important.*

Another space scale that is contained in $\mathrm{B}^{s,\mathrm{diff}}_{p,q}(\mathbb{R}^n)$ is that of Hölder continuous functions.

Lemma 2.12. *Let* $0 < s < 1$. *Then there holds*

$$\mathrm{B}^s_{\infty,\infty}(\mathbb{R}^n) \cong \mathrm{C}^{0,s}(\mathbb{R}^n).$$

Some readers might have already encountered the so-called *Sobolev-Slobodeckiĭ* spaces $\mathrm{W}^{s,p}(\mathbb{R}^n)$, which we briefly recall here:

Definition 2.13. *Let* $0 < s < 1$ *and* $1 \leqslant p < \infty$. *Given a measurable subset* Ω *of* $\mathbb{R}^n$, *the space* $\mathrm{W}^{s,p}(\Omega)$ *is defined as the linear space of all* $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ *such that*

$$\|u\|_{\mathrm{W}^{s,p}(\Omega)} := \|u\|_{\mathrm{L}^p(\Omega)} + \Big(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}(x,y)\Big)^{\frac{1}{p}}$$

is finite. The double integral quantity on the right-hand side of the previous equation is often called the (s,p)-Gagliardo seminorm *of* u.

The link to Besov spaces then is provided by

Theorem 2.14. *Let* $0 < s < 1$ *and* $1 \leqslant p < \infty$. *Then we have* $\mathrm{W}^{s,p}(\mathbb{R}^n) \cong \mathrm{B}^{s,\mathrm{diff}}_{p,p}(\mathbb{R}^n)$.

Proof. TBI. □

Lemma 2.15. *Let* $0 < s < 1$ *and* $1 \leqslant p, q < \infty$ *be such that* $sp < n$. *If* $r \in [1,\infty)$ *is such that there exists a constant* $c = c(n,s,p,q) > 0$ *with*

$$\|u\|_{\mathrm{L}^r(\mathbb{R}^n)} \leqslant c[u]_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)} \tag{2.20}$$

for all $u \in \mathrm{C}^\infty_c(\mathbb{R}^n)$, *then* $r = \frac{np}{n-sp}$.

Proof. Let $u \in \mathrm{C}^\infty_c(\mathbb{R}^n)\setminus\{0\}$ be arbitrary and consider, for $\lambda > 0$, the function $u_\lambda(x) := u(\lambda x)$. We then estimate, using a change of variables in the first step

$$\begin{aligned} \frac{1}{\lambda^{\frac{n}{r}}}\|u\|_{\mathrm{L}^r(\mathbb{R}^n)} = \|u_\lambda\|_{\mathrm{L}^r(\mathbb{R}^n)} & \overset{(2.20)}{\leqslant} c[u_\lambda]_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)} \\ & \leqslant c\Big(\sum_{i=1}^n \int_0^\infty \frac{\|\tau_{i,h}u_\lambda\|_{\mathrm{L}^p(\mathbb{R}^n)}^q}{h^{1+sq}}\,\mathrm{d}h\Big)^{\frac{1}{q}} = (*). \end{aligned}$$

To compute $(*)$, we first change variables in the integral defining the inner L^p-norm and then in the outer integral to obtain

$$\int_0^\infty \frac{\|\tau_{i,h}u_\lambda\|_{\mathrm{L}^p(\mathbb{R}^n)}^q}{h^{1+sq}}\,\mathrm{d}h = \frac{1}{\lambda^{n\frac{q}{p}}}\int_0^\infty \frac{\|\tau_{i,\lambda h}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q}{h^{1+sq}}\,\mathrm{d}h \overset{\widetilde{h}=\lambda h}{=} \lambda^{sq-n\frac{q}{p}}\int_0^\infty \frac{\|\tau_{i,\widetilde{h}}u\|_{\mathrm{L}^p(\mathbb{R}^n)}^q}{\widetilde{h}^{1+sp}}\,\mathrm{d}\widetilde{h}.$$

Going back to $(*)$, we thus get

$$\frac{1}{\lambda^{\frac{n}{r}}}\|u\|_{\mathrm{L}^r(\mathbb{R}^n)} \leqslant c\lambda^{s-\frac{n}{p}}[u]^q_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)} \text{ and so } \|u\|_{\mathrm{L}^r(\mathbb{R}^n)} \leqslant c\lambda^{s-\frac{n}{p}-\frac{n}{r}}[u]^q_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)}.$$

By considering the limit passages $\lambda \searrow 0$ and $\lambda \nearrow \infty$, the ultimate inequality readily gives a contradiction unless $s - \frac{n}{p} - \frac{n}{r} = 0$. This condition, however, is equivalent to $r = \frac{np}{n-sp}$ and thus settles the lemma. $\square$

Theorem 2.16 (Sobolev-type Embedding Theorem). *Let* $0 < s < 1$, $1 \leqslant p < \frac{n}{s}$ *and let* $1 \leqslant q \leqslant \frac{np}{n-sp}$. *Then there holds*

$$\mathrm{B}^{s}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathrm{L}^{\frac{np}{n-sp}}(\mathbb{R}^n). \tag{2.21}$$

2.3. **Fractional differentiability of minima.** We now come to our first regularity result, that we state for *convex variational*, x-dependent integrands. We already have alluded to this topic in the introduction to the present chapter, and hereafter let $\Omega \subset \mathbb{R}^n$ be an open and bounded Lipschitz subset of $\mathbb{R}^n$ and $F\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is an integrand which satisfies the following conditions: Given $1 < p < \infty$,

(H1) for each $x \in \Omega$, the partial map $\mathbb{R}^{N\times n} \ni z \mapsto F(x,z) \in \mathbb{R}$ is of class C^2.

(H2) F is of uniform p-growth: There exist $c_1, c_2, c_3 > 0$ such that for all $x \in \Omega$ and all $z \in \mathbb{R}^{N\times n}$ there holds
$$c_1|z|^p - c_2 \leqslant F(x,z) \leqslant c_3(1+|z|^p).$$

(H3) there exists $0 < \alpha < 1$ and a constant $c_4 > 0$ such that for all $x, x' \in \Omega$ and all $z \in \mathbb{R}^{N\times n}$ there holds
$$|D_zF(x,z) - D_zF(x',z)| \leqslant c_4|x-x'|^{\alpha}(1+|z|^2)^{\frac{p-1}{2}}.$$

(H4) F is p-strongly convex: There exist $c_5, c_6 > 0$ such that for all $x \in \Omega$ and all $z, \xi \in \mathbb{R}^{N\times n}$ there holds
$$c_5(1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \leqslant \langle D_z^2F(z)\xi, \xi\rangle \leqslant c_6(1+|z|^2)^{\frac{p-2}{2}}|\xi|^2.$$

(H5) There exists $c_7 > 0$ such that for all $x \in \Omega$ and all $z \in \mathbb{R}^{N\times n}$ there holds
$$|D_zF(x,z)| \leqslant c_7(1+|z|^2)^{\frac{p-1}{2}}.$$

Here, D_zF denotes the partial derivative of F in the z- (i.e., the ultimate) variable. To get an idea of how these conditions play together, we utilise the metaprinciple '*passing to one derivative reduces the growth bound by one*'. Also note that (H5) actually arises as a consequence of (H2) and (H4). We shall return to this aspect in higher generality when studying strongly quasiconvex problems. For future reference, however, we make the following

Remark 2.17 (On p-strong convexity). *The p-strong convexity as introduced in* (H4) *from above precisely amounts to requiring that the map*
$$z \mapsto F(x,z) - \ell(1+|z|^2)^{\frac{p}{2}}$$
is convex for some $\ell > 0$, *the lowest and highest eigenvalues of the corresponding Hessians being boundable* independently *of* x. *As the higher (fractional) differentiability of (local) minima will crucially hinge on assumption* (H4), *we emphasize that the* method is restricted to (strongly) convex integrands; *at present, no higher (fractional) differentiability theory is known for (strongly) quasiconvex integrands as briefly discussed in the introduction.*

We state the first main result of this chapter for *local minimisers*:

Definition 2.18 (Local minimiser)**.** *Let F satisfy the requirements of* (H1)–(H5) *from above. We then say that $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is a* local minimiser *provided with the convention*

$$\mathscr{F}[u;\omega] := \int_\omega F(x, Du)\,\mathrm{d}x$$

for Lipschitz subsets $\omega \Subset \Omega$ the following holds:

(a) *For each $\omega \Subset \Omega$ with Lipschitz boundary $\partial\omega$ there holds $\mathscr{F}[u;\omega] < \infty$ and*
(b) *for each such ω and all $\varphi \in \mathrm{C}^{1,p}_0(\omega;\mathbb{R}^N)$ there holds*

$$\mathscr{F}[u;\omega] \leqslant \mathscr{F}[u+\varphi;\omega].$$

Note that *every minimiser is a local minimiser* but not necessarily vice versa. We may now state the core result of the present section:

Theorem 2.19 (Mingione, 2003)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and suppose that $F\colon \Omega \times \mathbb{R}^{N\times n}$ satisfies assumptions* (H1)–(H5) *from above with $2 \leqslant p < \infty$. Then every local minimiser $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ satisfies*

$$Du \in \mathrm{B}^{2\alpha/p}_{p,\infty}(\omega;\mathbb{R}^{N\times n})$$

for any $\omega \Subset \Omega$. Here, $0 < \alpha < 1$ is the Hölder exponent dictating the smoothness of the x-dependence as specified in hypothesis (H3) *from above.*

Note that any local minimiser satisfies the Euler-Lagrange equation

$$\int_\Omega \langle D_z F(x, Du), D\varphi\rangle\,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N). \tag{2.22}$$

Lemma 2.20 (Auxiliary estimates, $2 \leqslant p < \infty$)**.** *Let $p \geqslant 2$. Then there exists $c_p > 0$ such that for all $z, z' \in \mathbb{R}^{N\times n}$ there holds*

$$c_p(1+|z|^2+|z'|^2)^{\frac{p-2}{2}} \leqslant \int_0^1 \big(1+|(1-t)z+tz'|^2\big)^{\frac{p-2}{2}}\,\mathrm{d}t.$$

Proof. We put $\kappa := (p-2)/2 \geqslant 0$. As the argument is symmetric in $z, z' \in \mathbb{R}^{N\times n}$, we may moreover assume that $|z'| \leqslant |z|$. We then estimate

$$\big(1+|z|^2+|z'|^2\big)^\kappa \leqslant \big(2+2|z|^2\big)^\kappa \leqslant 2^\kappa\big(1+|z|^2\big)^\kappa$$

Now, for any $0 \leqslant t \leqslant \frac{1}{4}$ there holds

$$\begin{aligned}|z|^2 &\leqslant |z-t(z-z')|^2 + 2t|z-t(z-z')||z-z'| + t^2|z-z'|^2\\ &\leqslant 2|z-t(z-z')|^2 + 2t^2|z-z'|^2\\ &\leqslant 2|z-t(z-z')|^2 + 8t^2|z|^2 \leqslant 2|z-t(z-z')|^2 + \frac{1}{2}|z|^2\end{aligned}$$

and thus

$$(1+|z|^2)^\kappa \leqslant 4^\kappa(1+|z-t(z-z')|^2)^\kappa.$$

We integrate the previous inequality with respect to $t \in [0, \frac{1}{4}]$ and find by Jensen's inequality

$$\big(1+|z|^2+|z'|^2\big)^\kappa \leqslant 2^\kappa\big(1+|z|^2\big)^\kappa \leqslant 8^\kappa \int_0^1 (1+|(1-t)z+tz'|^2)^\kappa\,\mathrm{d}t.$$

The proof is complete. □

Proof of Theorem 2.19. Let $x_0 \in \Omega$ and let $r > 0$ be such that $0 < r < \frac{\mathrm{dist}(x_0,\partial\Omega)}{4}$. We pick $\rho \in \mathrm{C}^1_c(\Omega;[0,1])$ such that $\mathbb{1}_{\mathrm{B}(x_0,r)} \leqslant \rho \leqslant \mathbb{1}_{\mathrm{B}(x_0,2r)}$ and $|\nabla\rho| \leqslant \frac{2}{r}$. For some arbitrary but fixed $i \in \{1,...,n\}$ and $h > 0$ sufficiently small, we put

$$\varphi := \tau^-_{i,h}(\rho^2\tau^+_{i,h}u).$$

This map belongs to $\mathrm{W}^{1,p}_c(\Omega;\mathbb{R}^N)$ and thus is an admissible competitor for the Euler-Lagrange equation (2.22). Similarly as in the smooth setup, we insert this specific choice of φ into (2.22) and obtain

$$\int_\Omega \langle \tau_{i,h}D_zF(x,Du), D(\rho^2\tau_{i,h}u)\rangle \,\mathrm{d}x = 0. \tag{2.23}$$

For notational simplicity, we put $\mathcal{A}(x,Du(x)) := D_zF(x,Du(x))$. We rewrite

$$\begin{aligned}\tau_{i,h}\mathcal{A}(x) &= \mathcal{A}(x+he_i, Du(x+he_i)) - \mathcal{A}(x,Du(x))\\ &= (\mathcal{A}(x+he_i,Du(x+he_i)) - \mathcal{A}(x+he_i,Du(x)))\\ &\quad + (\mathcal{A}(x+he_i,Du(x)) - \mathcal{A}(x,Du(x))) =: \mathcal{A}_1(x) + \mathcal{A}_2(x).\end{aligned}$$

Adopting this notation, (2.23) becomes

$$\begin{aligned}\int_\Omega \langle \mathcal{A}_1(x), \rho^2\tau_{i,h}Du\rangle \,\mathrm{d}x \leqslant & -\int_\Omega \langle \mathcal{A}_2(x),\rho^2\tau_{i,h}Du(x)\rangle\,\mathrm{d}x\\ & -\int_\Omega \langle \mathcal{A}_1(x), 2\rho D\rho\otimes\tau_{i,h}u(x)\rangle\,\mathrm{d}x\\ & -\int_\Omega \langle \mathcal{A}_2(x), 2\rho D\rho\otimes\tau_{i,h}u(x)\rangle\,\mathrm{d}x \Longleftrightarrow: \mathbf{I} \leqslant \mathbf{II}+\mathbf{III}+\mathbf{IV}.\end{aligned}$$

The underlying reason for this particular splitting is that the very left term yields a *sign*, and we now need to see how the right-hand side terms can be conveniently controlled. Moreover, it is at this stage where we distinguish between the growth regimes $1 < p < 2$ and $2 \leqslant p < \infty$. We firstly suppose that $p \geqslant 2 < \infty$.

Ad **I** and **III**. Here we start off by employing the fundamental theorem of calculus for $\mathscr{L}^n$-a.e. $x \in \mathrm{spt}(\rho)$ by

$$\begin{aligned}\mathcal{A}_1(x) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(x+he_i, Du(x)+t\tau_{i,h}Du(x))\,\mathrm{d}t\\ &= \int_0^1 D_z\mathcal{A}(x+he_i,Du(x)+t\tau_{i,h}Du(x))\cdot\tau_{i,h}Du(x)\,\mathrm{d}t\\ &= \int_0^1 D^2_zF(x+he_i,Du(x)+t\tau_{i,h}Du(x))\,\mathrm{d}t\cdot\tau_{i,h}Du(x)\\ &=: \mathbb{B}_{i,h,x}[\tau_{i,h}Du(x),-].\end{aligned}$$

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By our assumptions on F, $\mathbb{B}_{i,h,x} \in \mathbb{S}(\mathbb{R}^{N\times n})$ is uniformly elliptic in x and symmetric. Therefore, $\mathbb{B}_{i,h,x}$ satisfies a suitable variant of the Cauchy-Schwarz inequality.

In particular, we may record that **III** can be estimated by

$$\begin{aligned}\mathbf{III} &\leqslant \varepsilon \int_\Omega \mathbb{B}_{i,h,x}[\rho\tau_{i,h}Du(x),\rho\tau_{i,h}Du(x)]\,\mathrm{d}x \\ &\qquad + C(\varepsilon)\int_\Omega \mathbb{B}_{i,h,x}[2\rho D\rho\otimes\tau_{i,h}u, 2\rho D\rho\otimes\tau_{i,h}u]\,\mathrm{d}x \\ &\leqslant \varepsilon \int_\Omega \mathbb{B}_{i,h,x}[\rho\tau_{i,h}Du(x),\rho\tau_{i,h}Du(x)]\,\mathrm{d}x \\ &\qquad + C(\varepsilon)\int_\Omega\int_0^1 \big(1+|Du(x)+t\tau_{i,h}Du(x)|^2\big)^{\frac{p-2}{2}}|\rho D\rho\otimes\tau_{i,h}u|^2\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

Now, for $p\geqslant 2$, the map $s\mapsto(1+s^2)^{\frac{p-2}{2}}$ is monotonically increasing. Therefore, the last term can be estimated via

$$\begin{aligned}&\int_\Omega\int_0^1 \big(1+|Du(x)+t\tau_{i,h}Du(x)|^2\big)^{\frac{p-2}{2}}|\rho D\rho\otimes\tau_{i,h}u|^2\,\mathrm{d}x \\ &\qquad\leqslant C\int_\Omega\int_0^1 \big(1+|Du(x)|^2+|\tau_{i,h}Du(x)|^2\big)^{\frac{p-2}{2}}|\rho D\rho\otimes\tau_{i,h}u|^2\,\mathrm{d}t\,\mathrm{d}x \\ &\qquad= C|h|^2\int_\Omega \big(1+|Du(x)|^2+|\tau_{i,h}Du(x)|^2\big)^{\frac{p-2}{2}}|\rho D\rho\otimes\Delta_{i,h}u|^2\,\mathrm{d}x = (*).\end{aligned}$$

To deal with the ultimate term, we note that $\frac{2}{p}+\frac{p-2}{p}=1$ and thus $(*)$ can be estimated by Hölder's inequality via

$$\begin{aligned}(*) &\leqslant C|h|^2\Big(\int_{\mathrm{spt}(\rho)}\big(1+|Du(x)|^2+|\tau_{i,h}Du(x)|^2\big)^{\frac{p}{2}}\,\mathrm{d}x + C\int_\Omega |\rho D\rho\otimes\tau_{i,h}u|^p\,\mathrm{d}x\Big) \\ &\leqslant C|h|^2\int_{\mathrm{B}(x_0,3r)}(1+|Du|^p)\,\mathrm{d}x.\end{aligned}$$

We collect the estimates gathered so far and arrive at

$$\begin{aligned}\int_\Omega \mathbb{B}_{i,h,x}[\rho\tau_{i,h}Du(x),\rho\tau_{i,h}Du(x)]\,\mathrm{d}x &\leqslant \varepsilon\int_\Omega \mathbb{B}_{i,h,x}[\rho\tau_{i,h}Du(x),\rho\tau_{i,h}Du(x)]\,\mathrm{d}x \\ &\qquad + C(\varepsilon)|h|^2\int_{\mathrm{B}(x_0,3r)}(1+|Du|^p)\,\mathrm{d}x + \mathbf{II}+\mathbf{IV}.\end{aligned}$$

Choosing $0<\varepsilon<1$ conveniently small, we may then absorb the very first term on the right-hand side of the previous inequality into the left-hand side. In a next step, we employ the ellipticity bound from (H4) and Lemma 2.20 to obtain (recall that $p\geqslant 2$)

$$\begin{aligned}\widetilde{c}_p\int_\Omega(1+|Du(x)|^2+|Du(x+he_i)|^2)^{\frac{p-2}{2}}|\tau_{i,h}Du|^2\,\mathrm{d}x \\ &\leqslant c_5\int_\Omega\int_0^1\Big(1+|Du+t\tau_{i,h}Du|^2\Big)^{\frac{p-2}{2}}|\rho\,\tau_{i,h}Du|^2\,\mathrm{d}t\,\mathrm{d}x \\ &\leqslant C(\varepsilon)|h|^2\int_{\mathrm{B}(x_0,3r)}(1+|Du|^p)\,\mathrm{d}x + c\mathbf{II}+c\mathbf{IV}.\end{aligned}$$

It remains to give estimates for **II** and **IV**. As to **II**, we estimate

$$\begin{aligned}\mathbf{II} &\leqslant c|h|^\alpha \int_\Omega (1+|Du|^2)^{\frac{p-1}{2}} \rho^2 |\tau_{i,h} Du|\,\mathrm{d}x \\ &\leqslant c|h|^\alpha \int_\Omega (1+|Du|^2)^{\frac{1}{2}\left(\frac{p-2}{2}+\frac{p}{2}\right)} \rho^2 |\tau_{i,h} Du|\,\mathrm{d}x \\ &\leqslant \varepsilon \int_\Omega (1+|Du|^2)^{\frac{p-2}{2}} |\rho\tau_{i,h} Du|^2\,\mathrm{d}x + C(\varepsilon)|h|^{2\alpha} \int_\Omega (1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x\end{aligned}$$

To cope with **IV**, we employ the Hölder bound on the coefficients to obtain

$$\begin{aligned}\mathbf{IV} &\leqslant C \int_\Omega |h|^\alpha (1+|Du(x)|^2)^{\frac{p-1}{2}} |\rho D\rho \otimes \tau_{i,h} u|\,\mathrm{d}x \\ &\leqslant C|h|^{1+\alpha} \int_\Omega (1+|Du|^p)\,\mathrm{d}x.\end{aligned}$$

Now note that $0<\alpha<1$, and hence we have $|h|^{1+\alpha} \leqslant |h|^{2\alpha}$; note that we assume that $0<|h|<1$ throughout. At this stage, we absorb the remaining terms and find that

(2.24)
$$\int_\Omega (1+|Du(x)|^2+|Du(x+he_i)|^2)^{\frac{p-2}{2}} |\rho\tau_{i,h} Du|^2\,\mathrm{d}x \leqslant C|h|^{1+\alpha} \int_\Omega (1+|Du|^p)\,\mathrm{d}x.$$

Now, by $2 \leqslant p < \infty$,

$$\begin{aligned}|\tau_{i,h} Du|^p &= |\tau_{i,h} Du|^{p-2} |\tau_{i,h} Du|^2 \\ &\leqslant c(1+|Du(x)|^2+|Du(x+he_i)|^2)^{\frac{p-2}{2}} |\tau_{i,h} Du(x)|^2,\end{aligned}$$

and this implies the following Besov-Nikolskiĭ-type estimate on Du:

$$\sum_{i=1}^n \int_{\mathrm{B}(x_0,r)} |\tau_{h,i} Du|^p\,\mathrm{d}x \leqslant C|h|^{p\cdot\frac{2\alpha}{p}}.$$

Therefore, u is locally of class $\mathrm{B}^{2\alpha/p}_{p,\infty}$. □

Mingione's paper is entitled *'The singular set of solutions to non-differentiable elliptic systems'*, and we wish to understand the underlying reason for this terminology. Here we briefly digress and remind the reader of general $\mathrm{C}^{1,\alpha}$-Hölder regularity results for functionals of the form (1.1). In fact, as we shall see later in a more general context, under conditions (H1)–(H5), (local) minima of the associated variational integrals in general do not possess Hölder continuous first derivatives in the entire Ω but only on a *relatively open* subset $\Omega_0 \subset \Omega$ with $\mathscr{L}^n(\Omega \setminus \Omega_u) = 0$. This is a specific phenomenon that only emerges in the vectorial situation (i.e. $N>2$) and is referred to as *partial Hölder regularity of minima*. In what follows, we set $\Sigma_u := \Omega \setminus \Omega_u$ and call Σ_u the *singular set* of a given local minimiser $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$.

Example 2.21 (De Giorgi, Giusti & Miranda)**.** *Let $n \geqslant 3$. We define bilinear forms on $\mathbb{R}^{n\times n}$ by*

$$A_{ij}^{\kappa,\lambda}(n-2,n,x) := \delta_{\kappa\lambda}\delta_{ij} + \Big((n-2)\delta_{i\kappa} + n\frac{x_i x_\kappa}{|x|^2}\Big)\Big((n-2)\delta_{j\lambda} + n\frac{x_j x_\lambda}{|x|^2}\Big), \qquad x \neq 0,$$

$$\mathcal{A}_{n-2,n,x}[y,z] := \sum_{\kappa,\lambda,i,j=1}^{n} A_{ij}^{\kappa\lambda}(n-2,n,x) y_i^\kappa z_j^\lambda, \qquad y = (y_i^\kappa),\ z = (z_j^\lambda),$$

$$\widetilde{A}_{ij}^{\kappa,\lambda}(u) := \delta_{\kappa\lambda}\delta_{ij} + \Big(\delta_{i\kappa} + \frac{4}{n-2}\frac{u_i u_\kappa}{1+|u|^2}\Big)\Big(\delta_{j\lambda} + \frac{4}{n-2}\frac{u_j u_\lambda}{1+|u|^2}\Big), \qquad u \in \mathbb{R}^n,$$

$$\widetilde{\mathcal{A}}(u)[y,z] := \sum_{i,j,\kappa,\lambda=1}^{n} \widetilde{A}_{ij}^{\kappa,\lambda}(u) y_i^\kappa z_j^\lambda, \qquad y = (y_i^\kappa),\ z = (z_j^\lambda).$$

(a) (De Giorgi). *The function $u\colon \mathrm{B}(0,1) \to \mathbb{R}^n$ given by $u(x) = u_\alpha(x) := x/|x|^\alpha$ for $\alpha := (n/2)(1-((2n-2)^2+1)^{-1/2})$ belongs to* $\mathrm{W}^{1,2}(\mathrm{B}(0,1);\mathbb{R}^n)$ *and is an unbounded weak solution of the elliptic system* $\operatorname{div}(\mathcal{A}_{n-2,n,x}(\mathrm{D}u)) = 0$ *in* $\mathrm{B}(0,1)$.

(b) (Giusti & Miranda). *The function $u\colon$ $\mathrm{B}(0,1) \to \mathbb{R}^n$ given by $u(x) = x/|x|$ belongs to* $(\mathrm{W}^{1,2}\cap\mathrm{L}^\infty)(\mathrm{B}(0,1);\mathbb{R}^n)$ *and is a discontinuous weak solution of the elliptic system* $\operatorname{div}(\widetilde{\mathcal{A}}(u)\,\mathrm{D}u) = 0$ *in* $\mathrm{B}(0,1)$.

Even though $\mathscr{L}^n(\Sigma_u) = 0$ can be established in a variety of situations, Σ_u can be spread out very much. In this respect, one aims to quantify the size of Σ_u, and this is usually accomplished by estimating the Hausdorff dimension of Σ_u.

So, e.g., within the setup of (H1)–(H5) from above, we will prove that there exists a number $\widetilde{\alpha} = \widetilde{\alpha}(\alpha,n,p) \in (0,\alpha]$ and an open subset $\Omega_u \subset \Omega$ with $\mathscr{L}^n(\Omega\setminus\Omega_u) = 0$ such that $u \in \mathrm{C}^{1,\widetilde{\alpha}}(\Omega_0;\mathbb{R}^N)$. The singular set Σ_u moreover is contained in

$$\begin{aligned}\Sigma_0 \cup \Sigma_1 := &\Big\{x \in \Omega\colon\ \liminf_{r\searrow 0} \fint_{\mathrm{B}(x,r)} |Du - (Du)_{\mathrm{B}(x,r)}|^p \,\mathrm{d}y > 0\Big\} \\ &\cup \Big\{x \in \Omega\colon\ \limsup_{r\searrow 0} |(Du)_{\mathrm{B}(x,r)}| = \infty\Big\}.\end{aligned} \tag{2.25}$$

Deferring the precise discussion to a later point in the manuscript, we now aim to quantify the Hausdorff dimension of the singular set.

We briefly recall the requisite framework. Given $s \geqslant 0$ and $A \subset \mathbb{R}^n$, we define for $\delta > 0$

$$\mathscr{H}^s_\delta(A) := \inf\Big\{\sum_{j=1}^\infty \omega_s\Big(\frac{\operatorname{diam}(C_j)}{2}\Big)^s\colon\ A \subset \bigcup_{j=1}^\infty C_j,\ \operatorname{diam}(C_j) \leqslant \delta \text{ for all } j \in \mathbb{N}\Big\},$$

and note that $\delta \mapsto \mathscr{H}^s_\delta(A)$ is non-decreasing; we then define the outer Hausdorff measure $\mathscr{H}^s(A)$ by

$$\mathscr{H}^s(A) := \sup_{\delta>0} \mathscr{H}^s_\delta(A) = \lim_{\delta\searrow 0} \mathscr{H}^s_\delta(A).$$

As to the underlying limiting process, a nice terminology stems from the monograph of Evans & Gariepy, asserting that *sending $\delta \searrow 0$ forces to follow the local geometry of A*. We do not aim to give an overarching treatment of the (outer) Hausdorff measure, but confine to the following

Lemma 2.22. *If $0 \leqslant s < \infty$ and $A \subset \mathbb{R}^n$ satisfies $\mathscr{H}^s(A) < \infty$, then $\mathscr{H}^t(A) = 0$ for all $t > s$.*

Proof. Let $\varepsilon > 0$ be arbitrary. Given $\delta > 0$, let (C_j) be a cover of A with $\operatorname{diam}(C_j) \leqslant \delta$ for all $j \in \mathbb{N}$. Then we estimate

$$\mathscr{H}_\delta^{s+\varepsilon}(A) \leqslant C \sum_{j=1}^{\infty} \Big(\frac{\delta}{2}\Big)^{\varepsilon} \Big(\frac{\operatorname{diam}(C_j)}{2}\Big)^{s} \leqslant C\delta^{\varepsilon} \xrightarrow{\delta\searrow 0} 0.$$

Thus, $\mathscr{H}^t(A) = 0$ and the proof is complete. □

The preceding lemma motivates to introduce the notion of *Hausdorff dimension* as follows:

Definition 2.23 (Hausdorff dimension)**.** *Let $A \subset \mathbb{R}^n$. We then define the* Hausdorff dimension of A *by*

$$\dim_{\mathscr{H}}(A) := \inf\{s \geqslant 0\colon\ \mathscr{H}^s(A) = 0\}.$$

It is not too difficult to see that for any $A \subset \mathbb{R}^n$ there holds $\dim_{\mathscr{H}}(A) \leqslant n$. Strikingly, $\dim_{\mathscr{H}}$ can also attain fractional values; one may think of fractals such as the Koch snowflake.

Our principal aim is to bound the Hausdorff dimension based on the higher differentiability assertion from Theorem 2.19. For this, we require the following *measure density lemma* due to Giusti:

Lemma 2.24. *Let $A \subset \mathbb{R}^n$ be open and let μ be a finite Radon measure on A such that $\mu(A) < \infty$. For $0 < t < n$, we have $\dim_{\mathscr{H}}(A^t) \leqslant t$, where*

$$E^t := \Big\{x \in A\colon\ \limsup_{r\searrow 0} \frac{\mu(\mathrm{B}(x,r))}{r^t} > 0\Big\}.$$

For the proof of the lemma, we recall Vitali's covering lemma:

Lemma 2.25 (Vitali covering lemma)**.** *Let $\{\mathrm{B}_j\colon\ j \in \mathcal{J}\}$ be an arbitrary collection of (non-degenerate) balls in $\mathbb{R}^d$ such that*

$$\sup\Big\{\operatorname{rad}(\mathrm{B}_j)\colon\ j \in \mathcal{J}\Big\} < \infty,$$

$\operatorname{rad}(\mathrm{B})$ *denoting the radius of the ball* B*. Then there exists a countable subcollection $\{\mathrm{B}_j\colon\ j \in \mathcal{J}'\}$, $\mathcal{J}' \subset \mathcal{J}$, of balls from the original collection which are disjoint and satisfy*

$$\bigcup_{j\in\mathcal{J}} \mathrm{B}_j \subset \bigcup_{j\in\mathcal{J}'} 5\,\mathrm{B}_j,$$

where $5\,\mathrm{B}$ *is the ball with the same center as* B *but five times its radius.*

In the previous lemma, it is possible to replace the number 5 by any number $3 < t \leqslant 5$, but *not* 3, a fact that we do not need per se but is worth mentioning. We now may come to the

Proof of Lemma 2.24. Given $j \in \mathbb{N}$, we put

$$E_j^t := \Big\{x \in A\colon\ \limsup_{r\searrow 0} \frac{\mu(\mathrm{B}(x,r))}{r^t} > \frac{1}{j}\Big\}$$

and aim to show $\mathscr{H}^{t+\varepsilon}(E_j^t) = 0$ for all $j \in \mathbb{N}$. Then, since $E_t = \bigcup_{j\in\mathbb{N}} E_j^t$ and $\mathscr{H}^{t+\varepsilon}$ is an outer measure, $\mathscr{H}^{t+\varepsilon}(E^t) \leqslant \sum_{j\in\mathbb{N}} \mathscr{H}^{t+\varepsilon}(E_j^t) = 0$. Hence fix $j \in \mathbb{N}$, $\delta > 0$ and, for each $x \in E_j^t$, pick $r(x) \in (0,\delta)$ such that $\mu(\mathrm{B}(x, r(x))) > r(x)^t/j$. Then

$\{\mathrm{B}(x, r(x))\colon\ x \in E_j^t\}$ covers E_j^t, and we thus may apply Vitali's covering lemma from above to obtain a sequence $\{x_i\} \subset E_j^t$ such that $E_j^t \subset \bigcup_{i\in\mathbb{N}} \mathrm{B}(x_i, 5r(x_i))$ and the balls $\mathrm{B}(x_i, r(x_i))$ are pairwise disjoint. We then have for any $j \in \mathbb{N}$

$$\begin{aligned}\mathscr{H}_\delta^{t+\varepsilon}(E_j^t) &\leqslant \omega_{t+\varepsilon}\sum_{i=1}^\infty (5r(x_i))^{t+\varepsilon} \leqslant \omega_{t+\varepsilon}5^{t+\varepsilon}\delta^\varepsilon \sum_{i=1}^\infty r(x_i)^t \\ &\leqslant j\omega_{t+\varepsilon}5^{t+\varepsilon}\delta^\varepsilon \sum_{i=1}^\infty \mu(\mathrm{B}(x_i, r(x_i)) \leqslant j\omega_{t+\varepsilon}5^{t+\varepsilon}\delta^\varepsilon\mu(A).\end{aligned}$$

Now, as $\mu(A) < \infty$, the last term tends to zero as $\delta \searrow 0$. Therefore, $\mathscr{H}^{t+\varepsilon}(E_j^t) = 0$ and we conclude. The proof is complete. □

Remark 2.26. *The above proof remains valid provided the condition on μ to be a Radon measure is replaced by μ being a finite and countably superadditive set function.*

In order to apply Lemma 2.24, we wish to transfer the fractional differentiability information on Du to a size bound on $\dim_{\mathscr{H}}(\Sigma_u)$. For this we need a Poincaré-type inequality as follows:

Lemma 2.27. *Let $0 < s < 1$ and $1 \leqslant p < \infty$. Then there exists a constant $c > 0$ such that for all $u \in \mathrm{W}^{s,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ and all $R > 0$ there holds*

$$\fint_{\mathrm{B}(x_0,R)} |v - (v)_{\mathrm{B}(x_0,R)}|^p \,\mathrm{d}x \leqslant cR^{sp} \fint_{\mathrm{B}(x_0,R)}\int_{\mathrm{B}(x_0,R)} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y.$$

Proof. Let $u \in \mathrm{W}^{s,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$. We estimate

$$\fint_{\mathrm{B}(x_0,r)} |v(x) - (v)_{\mathrm{B}(x_0,r)}|^p \,\mathrm{d}x \leqslant \fint_{\mathrm{B}(x_0,r)}\fint_{\mathrm{B}(x_0,r)} |v(x)-v(y)|^p \,\mathrm{d}y\,\mathrm{d}x.$$

Now, if $x, y \in \mathrm{B}(x_0, r)$, then $|x-y| \leqslant 2r$ and so $|x-y|^{n+sp} \leqslant 2^{n+sp}r^{n+sp}$. In conclusion,

$$\fint_{\mathrm{B}(x_0,r)} |v(x) - (v)_{\mathrm{B}(x_0,r)}|^p \,\mathrm{d}x \leqslant \frac{2^{n+sp}r^{sp}}{\omega_n}\fint_{\mathrm{B}(x_0,r)}\int_{\mathrm{B}(x_0,r)} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y,$$

and this concludes the proof. □

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Now, given $0 < s < 1$ and $1 < p < \infty$, we put for a given $v \in \mathrm{W}^{s,p}(\mathbb{R}^n;\mathbb{R}^m)$

$$\mu_v(A) := \iint_{A\times A} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y, \qquad A \subset \mathbb{R}^n \text{ open}.$$

In this situation, Lemma 2.24 and the remark afterwards readily imply

$$\dim_{\mathscr{H}}\Big(\Big\{x \in \mathbb{R}^n\colon\ \limsup_{r\searrow 0}\frac{\mu_v(\mathrm{B}(x,r))}{r^{n-sp}} > 0\Big\}\Big) \leqslant n - sp.$$

Let $x_0 \in \Sigma_0$, the latter set being defined by (2.25). Given α as in (H3), we pick any $0 < \beta < 2\alpha/p$. By Theorem 2.19, $Du \in (\mathrm{B}^{2\alpha/p}_{p,\infty})_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$. Then, by Proposition 2.6 and $\mathrm{B}^s_{p,p} \simeq \mathrm{W}^{s,p}$, we obtain that

$$Du \in \mathrm{W}^{\beta,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n}). \tag{2.26}$$

For any compact subset ω of Ω with Lipschitz boundary $\partial\omega$, we have

$$\begin{aligned} x_0 \in \omega\cap\Sigma_0 \Longrightarrow 0 &< \liminf_{r\searrow 0} \fint_{\mathrm{B}(x_0,r)} |Du-(Du)_{x_0,r}|^p\,\mathrm{d}x \\ &\leqslant C\limsup_{r\searrow 0}\frac{1}{r^{n-\beta p}}\iint_{\mathrm{B}(x_0,r)\times\mathrm{B}(x_0,r)} \frac{|Du(x)-Du(y)|^p}{|x-y|^{n+\beta p}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

In conclusion, $\dim_{\mathscr{H}}(\omega\cap\Sigma_0)\leqslant n-sp$ and so, picking a sequence $(\omega_j)\subset\mathscr{P}(\Omega)$ of Lipschitz subsets with $\omega_j\nearrow\Omega$, $\dim_{\mathscr{H}}(\Sigma)\leqslant n-\beta p$ for any $\beta<2\alpha/p$, and thus $\dim_{\mathscr{H}}(\Sigma_0)\leqslant n-2\alpha$ by sending $\beta\nearrow\frac{2\alpha}{p}$.

Let us now see how $\dim_{\mathscr{H}}(\Sigma_1)$ can be controlled. We aim to show $\mathscr{H}^{n-2\beta+\varepsilon}(\Sigma_0)=0$ for any $\varepsilon>0$. To this end, fix $0<\varepsilon_0<\varepsilon$ and put

$$S := \Big\{x_0\in\Omega\colon\ \limsup_{r\searrow 0}\frac{\mu_u(\mathrm{B}(x_0,r))}{r^{n-2\beta+\varepsilon_0}}>0\Big\}.$$

As above, $\dim_{\mathscr{H}}(S)\leqslant n-2\beta$ and thus $\mathscr{H}^{n-2\beta+\varepsilon}(S)=0$. We establish

$$\Sigma_1\subset S \tag{2.27}$$

or, equivalently, $S^c\subset\Sigma_1^c$. The latter amounts to showing that

$$\limsup_{r\searrow 0}\frac{\mu_u(\mathrm{B}(x_0,r))}{r^{n-2\beta+\varepsilon_0}}=0\Longrightarrow\limsup_{r\searrow 0}|(Du)_{\mathrm{B}(x_0,r)}|<\infty. \tag{2.28}$$

Let $x_0\in S^c$ and let $R>0$ be arbitrary but small enough. In consequence,

$$\limsup_{k\to\infty}\frac{\mu_u(\mathrm{B}(x_0,2^{-k}r))}{(2^{-k}r)^{n-2\beta+\varepsilon_0}}\leqslant 1. \tag{2.29}$$

We then estimate, using Jensen's inequality in the first inequality,

$$\begin{aligned} |(Du)_{\mathrm{B}(x_0,2^{-k-1}R)}-(Du)_{\mathrm{B}(x_0,2^{-k}R)}|^p &\leqslant \fint_{\mathrm{B}(x_0,2^{-k-1}R)}|Du-(Du)_{\mathrm{B}(x_0,2^{-k}R)}|^p\,\mathrm{d}x \\ &\leqslant 2^n\fint_{\mathrm{B}(x_0,2^{-k}R)}|Du-(Du)_{\mathrm{B}(x_0,2^{-k}R)}|^p\,\mathrm{d}x \\ &= (*), \end{aligned}$$

and by the fractional Poincaré inequality, Lemma 2.27, together with the improved regularity $Du\in(\mathrm{B}^{\beta}_{p,p})_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$, it is then possible to bound $(*)$ by

$$\begin{aligned} (*) &\leqslant C(n,\beta,p)\Big(\frac{R}{2^k}\Big)^{2\beta-n}\mu_u(\mathrm{B}(x_0,2^{-k}R)) \\ &\leqslant C(n,\beta,p)\Big(\frac{R}{2^k}\Big)^{\varepsilon_0}\Big(\frac{R}{2^k}\Big)^{2\beta-n-\varepsilon_0}\mu_u(\mathrm{B}(x_0,2^{-k}R)) \\ &\leqslant C(n,\beta,p)\Big(\frac{1}{2^k}\Big)^{\varepsilon_0}. \end{aligned}$$

At this stage, we are in position to employ a geometric series and dyadic nesting argument to conclude that $\limsup_{r\searrow 0}|(Du)_{\mathrm{B}(x_0,r)}|<\infty$ hold.

We thus have established the following

Theorem 2.28 (Mingione, 2003)**.** *Let $2\leqslant p<\infty$. Subject to* (H1)–(H5)*, let $u\in\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimiser of the associated variational integral $v\mapsto\int F(x,Dv)\,\mathrm{d}x$. Defining Σ_u by* (2.25)*, there holds*

$$\dim_{\mathscr{H}}(\Sigma_u)\leqslant n-2\alpha. \tag{2.30}$$

In a next step, we briefly record the relevant modifications to make the above proof work in the superlinear, subquadratic growth situation, too; that is, we consider $1 < p < 2$. The starting point here is the following modification of Lemma 2.20:

Lemma 2.29 (Auxiliary estimates, $1 < p \leqslant 2$; Acerbi & Fusco 1989)**.** *For every $-\frac{1}{2} < \gamma < 0$ there exists constants $c_\gamma, C_\gamma > 0$ such that for every $\mu \geqslant 0$ there holds*

$$c_\gamma(\mu + |\xi|^2 + |\eta|^2)^\gamma \leqslant \int_0^1 \left(\mu^2 + |\eta + t(\xi - \eta)|^2\right)^\gamma \,\mathrm{d}t \leqslant C_\gamma(\mu + |\xi|^2 + |\eta|^2)^\gamma$$

for all $\xi, \eta \in \mathbb{R}^m$. Here we tacitly assume that $|\xi| \cdot |\eta| \neq 0$ provided $\mu = 0$.

Proof. For the left inequality, observe that for any $0 \leqslant t \leqslant 1$ there holds

$$(\mu^2 + |\xi + t(\xi - \eta)|^2)^{-\gamma} \leqslant 2^{-\gamma}(\mu^2 + |\xi|^2 + |\eta|^2)^{-\gamma}$$

for $\gamma < 0$. Rewriting and regrouping, an integration with respect to $t \in [0, 1]$ yields the desired left inequality. For the upper bound, we firstly establish that for all $a, b \geqslant 0$ there holds

$$\int_0^1 (a + tb)^{2\gamma} \,\mathrm{d}t \leqslant C(a^2 + b^2)^\gamma.$$

Suppose that $0 \leqslant b \leqslant a$ first. Then we conclude that $a^2 + b^2 \leqslant 2a^2$ and hence

$$\int_0^1 (a + tb)^{2\gamma} \,\mathrm{d}t \leqslant a^{2\gamma} \leqslant 2^{-\gamma}(a^2 + b^2)^\gamma.$$

If $0 \leqslant a < b$, then by a change of variables

$$\int_0^1 (a + tb)^{2\gamma} \,\mathrm{d}t \leqslant \frac{(a + b)^{2\gamma+1}}{(2\gamma + 1)b} \leqslant \frac{2}{2\gamma + 1}(a^2 + b^2)^\gamma.$$

To prove the desired inequality, we now assume that $|\xi| \leqslant |\eta|$ and $\eta \neq \xi$. Now let ξ_0 be the point on the line segment $[\xi, \eta]$ with least norm and put $s_0 := |\xi_0 - \eta|/|\xi - \eta|$. Also, for $t \in [0, 1]$, put

$$\varphi_\lambda(t) := (\mu^2 + |\eta + t(\lambda - \eta)|^2)^\gamma.$$

Rest TBI. □

We then have the following analogue of Theorem 2.19:

Theorem 2.30 (Higher Sobolev regularity in the subquadratic case)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and suppose that $F\colon \Omega \times \mathbb{R}^{N\times n}$ satisfies assumptions* (H1)–(H5) *from above with $1 < p < 2$. Then every local minimiser $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ satisfies*

$$Du \in \mathrm{B}^{(1+\alpha)/2}_{p,\infty}(\omega; \mathbb{R}^{N\times n})$$

for any $\omega \Subset \Omega$. Here, $0 < \alpha < 1$ is the Hölder exponent dictating the smoothness of the x-dependence as specified in hypothesis (H3) *from above.*

Sketch of proof. By Lemma 2.29, we equally arrive at the crucial estimation (2.24). We then obtain by Hölder's inequality

$$\int_\Omega \left(1 + |Du(x)|^2 + |Du(x + he_i)|^2\right)^{\frac{p-2}{2}} \left|\frac{\rho\tau_{i,h} Du(x)}{h^{\frac{1+\alpha}{2}}}\right|^2 \mathrm{d}x \leqslant C \int_\Omega (1 + |Du|^2)^{\frac{p}{2}} \,\mathrm{d}x.$$

Since $1 < p < 2$, we obtain

$$
\begin{aligned}
\int_\Omega \left|\frac{\rho\tau_{i,h}Du(x)}{h^{\frac{1+\alpha}{2}}}\right|^p \mathrm{d}x &= \int_\Omega \Big(\left|\frac{\rho\tau_{i,h}Du(x)}{h^{\frac{1+\alpha}{2}}}\right|^p (1+|Du(x)|^2+|Du(x+he_i)|^2)^{\frac{p(p-2)}{2}} \times \\
&\quad \times (1+|Du(x)|^2+|Du(x+he_i)|^2)^{-\frac{p(p-2)}{2}}\,\mathrm{d}x \\
&\leqslant c_p \int_\Omega \big(1+|Du(x)|^2+|Du(x+he_i)|^2\big)^{\frac{p-2}{2}} \left|\frac{\rho\tau_{i,h}Du(x)}{h^{\frac{1+\alpha}{2}}}\right|^2 \mathrm{d}x \\
&\quad + c_p \int_{\mathrm{spt}(\rho)} (1+|Du(x)|^2+|Du(x+he_i)|^2)^{-\frac{p(p-2)}{2}\cdot\frac{2}{2-p}}\,\mathrm{d}x \\
&\leqslant C_p \int_\Omega (1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x.
\end{aligned}
$$

Hence Du locally belongs to $\mathrm{B}^{(1+\alpha)/2}_{p,\infty}$, and the proof is complete. □

Before we come to generalisations, extensions and sharpenings of the foregoing theory, we make the following

Remark 2.31 (Singular set estimates in the strongly quasiconvex case)**.** *The key feature in the above approach of bounding the Hausdorff dimension of the singular set is* (H4)*, allowing us to employ higher fractional differentiability estimates. If F is merely assumed p-strongly quasiconvex, condition* (H4) *(which amounts to convexity of $F(x,\cdot)$ for each $x \in \Omega$) is* not *available. Thus, the method presented above does not apply to the quasiconvex situation, and in general, only one result seems to be available up to date, namely*

- Kristensen, J., Mingione, G.: The Singular Set of Lipschitzian Minima of Multiple Integrals. Arch. Rational Mech. Anal. 184 (2007) 341–369.

Still, as is displayed in the title, the previous reference only provides a result for Lipschitz minimisers; note the crucial difference between $\mathrm{C}^{0,1}$ *(Lipschitz continuity, boundedness of derivatives) and* $\mathrm{C}^{1,\alpha}$ *(Hölder continuity of derivatives). We will discuss the underlying approach later. The core idea here is to combine the concept of* set porosity *with so-called* Doronsorro estimates*, thereby leading to a geometric interpretation of suitable excess quantities.*

2.4. **A quick introduction to functions of bounded variation.** In the previous section we discussed a quantified compactness gain for variational problems in the p-growth framework, $1 < p < \infty$. We now turn our attention to the limiting case $p = 1$, and to this end, we need a different function space setup first.

To motivate our particular choice of function spaces, we reembark on the problem setting described in Section 1, cf. (1.2) subject to (p-growth), but now with $p = 1$. Here we put $\mathscr{D}_{u_0} := u_0 + \mathrm{W}^{1,1}_0(\Omega;\mathbb{R}^N)$, and observe that all steps of the direct method work *equally well* up to the point where we have to

extract a weakly convergent subsequence

from the particular considered minimising sequences $(u_j) \subset \mathscr{D}_{u_0}$. Note carefully that, as an L^1-based space, $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ fails to be reflexive. Hence, the Banach-Alaoglu-Bourbaki theorem does *not* give us the requisite weak precompactness result. To *enforce* compactness, it thus stands to reason to consider weaker topologies; then, as a drawback, we are lead to consider a larger space on which the original functional is not even defined at all. We shall therefore be forced to *extend* the original functional (1.2) to the larger space just in order to come up with a *adequate notion of minimality.*

Let us now see what could be a reasonable candidate for such a *larger* function space.

We begin by recalling that every $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$, $m \in \mathbb{N}$, induces a distribution $T_u \in \mathscr{D}'(\mathbb{R}^n;\mathbb{R}^m)$ via

$$\langle T_u, \varphi\rangle := \int_{\mathbb{R}^n} u\varphi \,\mathrm{d}x, \qquad \varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^m).$$

April 15, 2019

For the following, we quickly recap some basic terminology from measure theory. Based on the anticipated weak*-compactness principle for Radon measures, we *employ a duality result.* Duality works for vector spaces, and the Radon measures with target $[0,\infty]$ do *not* form a vector space. We are thus bound to suitably extend this definition, and in doing so, we follow the approach due to Ambrosio, Fusco & Pallara.

Definition 2.32 (Measures)**.** *Let (X,Σ) be a measurable space and let $m \in \mathbb{N}$. A set function $\mu\colon \Sigma \to \mathbb{R}^m$ is called a* measure *provided $\mu(\varnothing) = 0$ and if $(E_j) \subset \Sigma$ is a sequence of pairwise disjoint sets, then*

$$\mu\Big(\bigcup_j E_j\Big) = \sum_j \mu(E_j).$$

Radon measures now can be introduced as follows.

Definition 2.33 (Radon measures)**.** *Let (X,d) be a locally compact, separable metric space. An $\mathbb{R}^m$-valued set function μ that is defined on the Borel σ-algebra of the relatively compact Borel subsets of X which is a measure on $(K,\mathscr{B}(K))$ for any compact $K \subset X$ is called a* Radon measure*. If, moreover, $|\mu|(X) < \infty$, then μ is called a* finite Radon measure*. The space of $\mathbb{R}^m$-valued, finite Radon measures is denoted $\mathcal{M}(X;\mathbb{R}^m)$.*

If $\mu = (\mu_1,...,\mu_m) \in \mathcal{M}(X;\mathbb{R}^m)$, we declare its total variation $|\mu|(X)$ via $|\mu| := |(|\mu_1|(X),...,|\mu_m(X)|)|$. It is important to note that in the situation of the preceding definition, $(\mathcal{M}(X), |\cdot|)$ forms a *normed space*, and by the following fundamental representation theorem, even a Banach space. The essential outcome can be conveniently rewritten as

$$\boxed{\mathcal{M}(X;\mathbb{R}^m) \cong (\mathrm{C}_0(X;\mathbb{R}^m))^*.}$$

Theorem 2.34 (Riesz representation theorem)**.** *Let (X,d) be a locally compact, separable metric space and suppose that $\Phi\colon\ \mathrm{C}_0(X;\mathbb{R}^m) \to \mathbb{R}$ is additive and bounded. That is, for all $x,y \in X$ there holds $\Phi(x+y) = \Phi(x) + \Phi(y)$, and*

$$\|\Phi\| := \sup\big\{|\Phi(f)|\colon\ f \in \mathrm{C}_0(X;\mathbb{R}^m),\ |f| \leqslant 1\big\}.$$

Then there exists a unique $\mathbb{R}^m$-valued, finite Radon measure $\mu = (\mu_1,...,\mu_m) \in \mathcal{M}(X;\mathbb{R}^m)$ such that

$$\Phi(f) = \sum_{k=1}^m \int_X f_k \,\mathrm{d}\mu_k \qquad \text{for all } f = (f_1,...,f_m) \in \mathrm{C}_0(X;\mathbb{R}^m).$$

Moreover, with the total variation from above, we have $\|\Phi\| = |\mu|(X)$.

We are now ready to give a dual characterisation of the total variation of Du for $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$:

$$
\begin{aligned}
|Du|(\Omega) &= \sup\Big\{ \int_\Omega \varphi \,\mathrm{d}Du \colon\; \varphi \in \mathrm{C}_0(\Omega;\mathbb{R}^N),\; |\varphi| \leqslant 1\Big\} \\
&= \sup\Big\{ \int_\Omega \varphi \,\mathrm{d}Du \colon\; \varphi \in \mathrm{C}_c^\infty(\Omega;\mathbb{R}^N),\; |\varphi| \leqslant 1\Big\} \\
&= \sup\Big\{ \int_\Omega u\cdot \mathrm{div}(\varphi) \,\mathrm{d}x \colon\; \varphi \in \mathrm{C}_c^\infty(\Omega;\mathbb{R}^N),\; |\varphi| \leqslant 1\Big\} \\
&= \sup\Big\{ \int_\Omega u\cdot \mathrm{div}(\varphi) \,\mathrm{d}x \colon\; \varphi \in \mathrm{C}_c^1(\Omega;\mathbb{R}^N),\; |\varphi| \leqslant 1\Big\}.
\end{aligned} \tag{2.31}
$$

Here, the first equality is due to Theorem 2.34. For the second and fourth equalities, we use that the closure of $\mathrm{C}_c^\infty(\Omega;\mathbb{R}^N)$ and $\mathrm{C}_c^1(\Omega;\mathbb{R}^N)$ with respect to the supremum norm is $\mathrm{C}_0(\Omega;\mathbb{R}^N)$; the third equality is just a reformulation of Du being the measure regular representative for the distributional derivate of u. Next, we record

Lemma 2.35. *Let $\Omega \subset \mathbb{R}^n$ be open and let $u, u_1, u_2, \ldots \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ such that $u_j \to u$ in $\mathrm{L}^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, that is, for every $K \Subset \Omega$ there holds $\|u_j - u\|_{\mathrm{L}^1(K;\mathbb{R}^N)} \to 0$ as $j \to \infty$. Then there holds*

$$|Du|(\Omega) \leqslant \liminf_{j\to\infty} |Du_j|(\Omega).$$

Proof. We work from the dual characterisation of the total variation, cf. (2.31). Let $\varphi \in \mathrm{C}_c^1(\Omega;\mathbb{R}^N)$ be arbitrary with $|\varphi| \leqslant 1$. Then we have

$$
\begin{aligned}
\int_\Omega u\cdot \mathrm{div}(\varphi)\,\mathrm{d}x &= \int_\Omega (u-u_j)\cdot \mathrm{div}(\varphi)\,\mathrm{d}x + \int_\Omega u_j\cdot \mathrm{div}(\varphi)\,\mathrm{d}x \\
&\leqslant \int_{\mathrm{spt}(\varphi)} |u-u_j|\,\mathrm{d}x\, \|\mathrm{div}(\varphi)\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^N)} + |Du_j|(\Omega).
\end{aligned}
$$

Now send $j \to \infty$ and pass to the supremum over all admissible φ to conclude. The proof is complete. ☐

Our original objective, namely to find a suitable topology on BV to yield good compactness results, now suggests to consider the following notions of convergence. Let $u, u_1, u_2, \ldots \in \mathrm{BV}(\Omega;\mathbb{R}^N)$. We say that

- (u_j) converges to u in the *norm topology* provided $\|u - u_j\|_{\mathrm{BV}(\Omega)} \to 0$ as $j \to \infty$. Here, we have set $\|v\|_{\mathrm{BV}(\Omega)} := \|u\|_{\mathrm{L}^1} + |Dv|(\Omega)$.
- (u_j) converges to u in the *strict sense* provided $d_s(u_j, u) \to 0$ as $j \to \infty$, where

$$d_s(u,v) := \|u-v\|_{\mathrm{L}^1(\Omega)} + \big|\,|Du|(\Omega) - |Dv|(\Omega)\big|$$

  is the *strict metric*.
- (u_j) converges to u in the *weak*-sense* provided $\|u_j - u\|_{\mathrm{L}^1(\Omega)} \to 0$ and $Du_j \overset{*}{\rightharpoonup} Du$ in the weak*-sense of Radon measures on Ω as $j \to \infty$.

Let us now discuss both the benefits and the drawbacks of each of these convergences. These notions of convergence are linked by the following diagram:

Norm convergence $\Rightarrow$ Strict convergence $\Rightarrow$ Weak*-convergence.

(i) *The norm topology.* When being endowed with the norm $\|\cdot\|_{\mathrm{BV}(\Omega)}$, $\mathrm{BV}(\Omega)$ becomes a Banach space. In fact, if (u_j) is a Cauchy sequence for $\|\cdot\|_{\mathrm{BV}(\Omega)}$, we firstly conclude that (u_j) is an L^1-Cauchy sequence. Since $\mathrm{L}^1(\Omega;\mathbb{R}^N)$ is Banach, we find $u\in\mathrm{L}^1(\Omega;\mathbb{R}^N)$ such that $u_j\to u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^N)$, and by Lemma 2.35, $|Du|(\Omega)\leqslant\liminf_{j\to\infty}|Du_j|(\Omega)$ so that $u\in\mathrm{BV}(\Omega;\mathbb{R}^N)$. On the other hand, by the BV-Cauchy property, given $\varepsilon>0$ there exists $l\in\mathbb{N}$ such that for all $j,i\geqslant l$ there holds

$$|D(u_j-u_i)|(\Omega)<\varepsilon.$$

If $i\geqslant l$ is fixed, $u_j-u_i\to u-u_i$ in $\mathrm{L}^1(\Omega;\mathbb{R}^N)$ implies by virtue of Lemma 2.35 that

$$|D(u-u_i)|(\Omega)\leqslant\liminf_{j\to\infty}|D(u_j-u_i)|(\Omega)<\varepsilon.$$

Hence, $u_j\to u$ with respect to $\|\cdot\|_{\mathrm{BV}(\Omega)}$, and the proof of the Banach space property is complete.

A key requirement on a reasonable notion of convergence on a function space is that it allows smooth approximation. This is not the case for norm convergence. In fact, pick $u\in(\mathrm{BV}\setminus\mathrm{W}^{1,1})(\Omega)$. In this situation, smooth approximability means that there exists $(u_j)\subset(\mathrm{BV}\cap\mathrm{C}^\infty)(\Omega)$ such that $\|u-u_j\|_{\mathrm{BV}(\Omega)}\to 0$. If this were possible, we would infer that (u_j) is $\|\cdot\|_{\mathrm{BV}}$-Cauchy; but $u_j\in(\mathrm{BV}\cap\mathrm{C}^\infty)(\Omega)=(\mathrm{W}^{1,1}\cap\mathrm{C}^\infty)(\Omega)$, and the BV- and $\mathrm{W}^{1,1}$-norms coincide on $\mathrm{W}^{1,1}(\Omega)$. So (u_j) would be a Cauchy-sequence in $\mathrm{W}^{1,1}(\Omega)$ and, since the latter is a Banach space, converge to some $v\in\mathrm{W}^{1,1}(\Omega)$. Clearly, $u=v$, and so we would obtain the contradictory $\mathrm{BV}(\Omega)=\mathrm{W}^{1,1}(\Omega)$ *which is easily seen to be false.* Moreover, the norm topology certainly cannot yield any desirable compactness result as described above. This is pretty much the same as for any infinite dimensional normed space, and so there is no hope to invoke the norm topology to play a crucial role in view of solving the variational principle (1.2) with $p=1$.

(ii) *Weak*-convergence.* By the Banach-Alaoglu-Bourbaki theorem in conjunction with the specific form of the Riesz representation theorem, Theorem 2.34, weak*-convergence is *designed* to yield good compactness results. This will eventually lead to minimising sequences possessing weak*-convergent subsequences by Theorem 2.37 below. However, another facet of the direct method is to make sure that the corresponding (weak*-)limit attains the correct traces; by this we make sure that the limit is an admissible competitor.

In view of this issue, we can anticipate right now that it is difficult to set up a reasonable trace theory compatible with weak*-convergence. To emphasize this issue, let us argue that

$$\overline{\mathrm{C}_c^\infty(\Omega;\mathbb{R}^N)}^{\mathrm{w}^*}=\mathrm{BV}(\Omega;\mathbb{R}^N),\tag{2.32}$$

where $\overline{\cdot}^{\mathrm{w}^*}$ denotes the closure with respect to weak*-convergence in the sense specified above. The space $\mathrm{BV}(\Omega;\mathbb{R}^N)$ is closed with respect to weak*-convergence, establishing '$\subseteq$'. For the reverse inclusion, let $u\in\mathrm{BV}(\Omega;\mathbb{R}^N)$ and pick, for given $j\in\mathbb{N}$, $\rho_j\in\mathrm{C}_c^\infty(\Omega;[0,1])$ with

$$\mathbb{1}_{\{\mathrm{dist}(\cdot,\partial\Omega)>\frac{2}{j}\}}\leqslant\rho_j\leqslant\mathbb{1}_{\{\mathrm{dist}(\cdot,\partial\Omega)>\frac{1}{j}\}}.$$

Convergence	Smooth Appr.	Seq. Comp.	Continuity of Tr
Norm convergence	–	–	✓
Strict convergence	✓	–	✓
Weak*-convergence	✓	✓	–

FIGURE 2. None of the convergences presented above meets all three requirements: Smooth approximation, sequential compactness of the closed unit ball (with respect to $\|\cdot\|_{\mathrm{BV}}$ and continuity of the (boundary) trace operator.

Now consider $v_j := \varphi_{1/j} * (\rho_j u)$. Clearly, $v_j \to u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^N)$. Now let $\varphi \in \mathrm{C}_c^\infty(\Omega;\mathbb{R}^{N\times n})$. We then estimate

$$\begin{aligned}\int_\Omega \varphi\,\mathrm{d}Dv_j &= \int_\Omega (\varphi_{1/j} * D(\rho_j u))\cdot\varphi\,\mathrm{d}x \\ &= \int_\Omega \big(u\otimes\nabla\rho_j\mathscr{L}^n\big)(\varphi_{1/j}*\varphi) + \int_\Omega \big(\rho_j Du\big)(\varphi_{1/j}*\varphi)\,\mathrm{d}x =: \mathbf{I}_j + \mathbf{II}_j.\end{aligned}$$

The integral $\mathbf{I}_j$ vanishes for large j as φ has compact support, and so $\mathrm{spt}(\rho_j)\cap \mathrm{spt}(\varphi_{1/j}*\varphi) = \varnothing$ for j sufficiently large. On the other hand, $\rho_j(\varphi_{1/j}*\varphi)\to\varphi$ in $\mathrm{C}_0(\Omega;\mathbb{R}^{N\times n})$. Therefore,

$$\int_\Omega \varphi\,\mathrm{d}Dv_j \to \int_\Omega \varphi\,\mathrm{d}Du, \qquad j\to\infty,$$

and hence (v_j) converges to u in the weak*-sense on $\mathrm{BV}(\Omega;\mathbb{R}^N)$. We note that (2.32) implies that the weak*-convergence cannot respect traces in any reasonable way; in particular, no reasonable notion of trace is continuous with respect to weak*-convergence. Hence, even though weak*-convergence proves useful in view of compactness results, we shall face severe issues to ensure that weak*-limits of suitable minimising sequences belong to the correct Dirichlet classes.

(iii) *Strict convergence.* A chief issue with respect to weak*-convergence is that – on the level of the gradients – mass can escape to the boundary $\partial\Omega$ in the limit. In order to prevent this phenomenon, *strict convergence forces* – by its very definition – *the masses to converge.* Even though strict convergence is too strong to yield good compactness results, it allows smooth approximation – see Theorem 2.36. We will also see, cf. Theorem 2.48 below, that the trace operator on BV is continuous with respect to strict convergence.

Theorem 2.36 (Strict smooth approximation)**.** *Let $u\in\mathrm{BV}(\Omega;\mathbb{R}^N)$. Then there exists a sequence $(u_j)\subset(\mathrm{C}^\infty\cap\mathrm{BV})(\Omega;\mathbb{R}^N)$ such that*

$$u_j \to u \qquad \textit{strictly in } \mathrm{BV}(\Omega;\mathbb{R}^N).$$

Proof. The key to almost all smooth approximation results is mollification. Here we do not necessarily work on the entire $\mathbb{R}^n$ and thus follow the usual scheme to *firstly localise, secondly mollify* and *finally patch together*. To this end, let $\varepsilon>0$ be arbitrary.

Step 1. Constructing the smooth approximation. We choose $m \in \mathbb{N}$ so large such that with

$$\Omega_k := \big\{x \in \Omega\colon\ \operatorname{dist}(x, \partial\Omega) > \frac{1}{m+k}\big\} \cap \mathrm{B}(0, m+k)$$

there holds

$$|Du|(\Omega \setminus \Omega_1) < \frac{\varepsilon}{4}. \tag{2.33}$$

We then put $U_0 := \varnothing$ and inductively define $U_k := \Omega_{k+1} \setminus \overline{\Omega_{k-1}}$ for $k \in \mathbb{N}_{\geqslant 1}$. For future reference, we remark that by construction, at most three U_k's overlap each.

In a next step, let (ρ_k) be a partition of unity subordinate to $(U_k)_{k\in\mathbb{N}}$. By this we understand that

(P1) $\rho_k \in \mathrm{C}_c^\infty(U_k; [0,1])$ for all $k \in \mathbb{N}$ and
(P2) $\sum_k \rho_k \equiv 1$ in Ω.

For each $k \in \mathbb{N}$, we pick $\varepsilon_k \in (0,1)$ such that $\operatorname{spt}(\varphi_{\varepsilon_k} * (\rho_k u)) \subset U_k$,

$$\int_\Omega |\varphi_{\varepsilon_k} * (\rho_k u) - \rho_k u| \,\mathrm{d}x < \frac{\varepsilon}{2^{k+2}}, \tag{2.34}$$

together with

$$\int_\Omega |\varphi_{\varepsilon_k} * (u \otimes \nabla\rho_k) - (u \otimes \nabla\rho_k)| \,\mathrm{d}x < \frac{\varepsilon}{2^{k+3}}. \tag{2.35}$$

Our candidate for the requisite smooth approximation then is given by

$$u_\varepsilon := \sum_{k=1}^\infty \varphi_{\varepsilon_k} * (\rho_k u).$$

Note that this is a *locally finite* sum: For each $x \in \Omega$ there exists a neighbourhood U such that only finitely many (namely, three) summands in the infinite sum defining u_ε actually contribute to $u_\varepsilon(y)$ for all $y \in U$. Since each of the summands is clearly of class C^∞, we have $u_\varepsilon \in \mathrm{C}^\infty(\Omega; \mathbb{R}^N)$.

Step 2. The L^1*-part.* We recall (P2) from above to find

$$\begin{aligned} \|u - u_\varepsilon\|_{\mathrm{L}^1(\Omega;\mathbb{R}^N)} &= \int_\Omega \left| (\sum_{k=1}^\infty \rho_k) u - \sum_{k=1}^\infty \varphi_{\varepsilon_k} * (\rho_k u) \right| \mathrm{d}x \\ &\leqslant \sum_{k=1}^\infty \int_{U_k} |\rho_k u - \varphi_{\varepsilon_k} * (\rho_k u)| \,\mathrm{d}x < \frac{\varepsilon}{4}. \end{aligned}$$

Thus $u_\varepsilon \to u$ in $\mathrm{L}^1(\Omega; \mathbb{R}^N)$ as $\varepsilon \searrow 0$.

Step 3. The total variation part. As established in step 2, $u_\varepsilon \to u$ in $\mathrm{L}^1(\Omega; \mathbb{R}^N)$ as $\varepsilon \searrow 0$ and thus, by Lemma 2.35, $|Du|(\Omega) \leqslant \liminf_{\varepsilon\searrow 0} |Du_\varepsilon|(\Omega)$. We thus must show that $\liminf_{\varepsilon\searrow 0} |Du_\varepsilon|(\Omega) \leqslant |Du|(\Omega)$ to conclude the proof. To this end, we first recall the equality

$$\int_{\mathbb{R}^n} (f * g) h \,\mathrm{d}x = \int_{\mathbb{R}^n} f (g * h) \,\mathrm{d}x \tag{2.36}$$

for all f, g, h. Aiming to employ the dual characterisation (2.31) of the total variation, we let $\varphi \in \mathrm{C}_c^1(\Omega;\mathbb{R}^{N\times n})$ with $|\varphi| \leqslant 1$ be arbitrary. We then rewrite

$$\begin{aligned}\int_\Omega u_\varepsilon \operatorname{div}(\varphi)\,\mathrm{d}x &= \sum_{k=1}^\infty \int_\Omega (\varphi_{\varepsilon_k} * (\rho_k u)) \operatorname{div}(\varphi)\,\mathrm{d}x \\ &= \sum_{k=1}^\infty \int_\Omega (\rho_k u) \operatorname{div}(\varphi_{\varepsilon_k} * \varphi)\,\mathrm{d}x \\ &= \sum_{k=1}^\infty \int_\Omega u \operatorname{div}(\rho_k \varphi_{\varepsilon_k} * \varphi)\,\mathrm{d}x - \int_\Omega (u\otimes \nabla\rho_k)(\varphi_{\varepsilon_k} * \varphi)\,\mathrm{d}x) = (*).\end{aligned}$$

By (P2), $\sum_k \nabla\rho_k = \nabla\sum_k \rho_k = 0$ in Ω and thus $\sum_k u\otimes\nabla\rho_k = 0$ in Ω. Therefore,

$$\begin{aligned}(*) &= \sum_{k=1}^\infty \int_\Omega u \operatorname{div}(\rho_k \varphi_{\varepsilon_k} * \varphi)\,\mathrm{d}x - \int_\Omega (u\otimes \nabla\rho_k)(\varphi_{\varepsilon_k} * \varphi)\,\mathrm{d}x \\ &= \sum_{k=1}^\infty \int_\Omega u \operatorname{div}(\rho_k \varphi_{\varepsilon_k} * \varphi)\,\mathrm{d}x - \int_\Omega (u\otimes \nabla\rho_k)(\varphi_{\varepsilon_k} * \varphi - \varphi)\,\mathrm{d}x = \mathbf{I} + \mathbf{II},\end{aligned}$$

with an obvious definition of $\mathbf{I}$ and $\mathbf{II}$.

The map $\rho_1(\varphi_{\varepsilon_1} * \varphi)$ is compactly supported in Ω and satisfies $|\rho_1(\varphi_{\varepsilon_1} * \varphi)| \leqslant 1$. Since at most three U_k's overlap each, we thus obtain

$$\begin{aligned}\mathbf{I} &\leqslant \int_\Omega u \operatorname{div}(\rho_1 \varphi_{\varepsilon_1} * \varphi)\,\mathrm{d}x + \sum_{k=2}^\infty \int_\Omega u \operatorname{div}(\rho_k \varphi_{\varepsilon_k} * \varphi)\,\mathrm{d}x \\ &\leqslant |Du|(\Omega) + \sum_{k=2}^\infty \int_\Omega u \operatorname{div}(\rho_k \varphi_{\varepsilon_k} * \varphi)\,\mathrm{d}x \\ &\leqslant |Du|(\Omega) + \frac{3}{4}\varepsilon,\end{aligned}$$

where we have used the dual characterisation of the total variation, cf. (2.31), in the second and assumption (2.33) in the third step. Ad $\mathbf{II}$. Arguing similarly as above, cf. (2.36), we find by $|\varphi| \leqslant 1$

$$\begin{aligned}\mathbf{II} &= \sum_{k=1}^\infty \left| \int_\Omega (u\otimes\nabla\rho_k)(\varphi_{\varepsilon_k} * \varphi - \varphi)\,\mathrm{d}x \right| \\ &\leqslant \sum_{k=1}^\infty |\varphi_{\varepsilon_k} * (u\otimes\nabla\rho_k) - u\otimes\nabla\rho_k|\,\mathrm{d}x \leqslant \frac{\varepsilon}{8}\sum_{k=1}^\infty 2^{-k} = \frac{\varepsilon}{8}.\end{aligned}$$

Summarising, since the estimates on $\mathbf{I}, \mathbf{II}$ do not depend on the specific choice of φ,

$$|Du_\varepsilon|(\Omega) \leqslant |Du|(\Omega) + \varepsilon.$$

Now send $\varepsilon \searrow 0$ to conclude. The proof is complete. □

Theorem 2.37 (Weak*-compactness theorem in BV)**.** *Let $\Omega \subset \mathbb{R}^N$ be open and bounded with Lipschitz boundary. If $(u_j) \subset \mathrm{BV}(\Omega;\mathbb{R}^N)$ is a sequence which is norm bounded in $\mathrm{BV}(\Omega;\mathbb{R}^N)$, then there exists $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ and a subsequence $(u_{j(i)}) \subset (u_j)$ such that*

$$u_{j(i)} \overset{*}{\rightharpoonup} u \qquad \text{in } \mathrm{BV}(\Omega;\mathbb{R}^N).$$

Proof. Let $(u_j) \subset \mathrm{BV}(\Omega;\mathbb{R}^N)$ be as in the theorem. Our strategy is to firstly approximate each u_j by some $v_j \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ in the strict metric. Then we extend v_j – employing the extension theorem for $\mathrm{W}^{1,1}$-maps – to some $Ev_j \in \mathrm{W}^{1,1}_c(\mathrm{B}(0,R);\mathbb{R}^N)$. Then we can conveniently apply the Riesz-Kolmogorov criterion, cf. Lemma 1.10, and obtain a subsequence for which the L^1-limit exists. We then have to establish that the restriction of this weak*-limit u belongs to BV and that the claimed weak*-convergence holds. We thus proceed in three steps.

Step 1. Smooth approximation and extension. For each $j \in \mathbb{N}$ we employ Theorem 2.36 to find $v_j \in (\mathrm{C}^\infty \cap \mathrm{BV})(\Omega;\mathbb{R}^N) = (\mathrm{C}^\infty \cap \mathrm{W}^{1,1})(\Omega;\mathbb{R}^N)$ such that

$$\|u_j - v_j\|_{\mathrm{L}^1(\Omega;\mathbb{R}^N)} < \frac{1}{j} \quad \text{and} \quad \big|\,|Du_j|(\Omega) - |Dv_j|(\Omega)\big| < \frac{1}{j}. \tag{2.37}$$

Every v_j thus belongs to $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^n)$ and thus we can employ the following extension theorem for $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$-maps: *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then there exists a bounded linear extension operator $E\colon \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N) \to \mathrm{W}^{1,1}(\mathbb{R}^n;\mathbb{R}^N)$. That is, for $Ev|_\Omega = v$ for any $v \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ and we have $\|Ev\|_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \leqslant C\|v\|_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)}$ for all $v \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$.*

We can always assume that the extension operator actually maps $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N) \to \mathrm{W}^{1,1}_c(\mathrm{B}(0,R);\mathbb{R}^N)$, where $R > 0$ is sufficiently large. In fact, we find $R > 0$ such that $\Omega \subset \mathrm{B}(0,R)$. We then pick $\rho \in \mathrm{C}^\infty_c(\mathbb{R}^n;[0,1])$ such that $\mathbb{1}_{\mathrm{B}(0,R)} \leqslant \rho \leqslant \mathbb{1}_{\mathrm{B}(0,2R)}$ and set $\widetilde{E}v := \rho Ev$.

Step 2. Precompactness of the extended sequence. Next consider $\widetilde{v}_j := Ev_j \in \mathrm{W}^{1,1}_c(\mathrm{B}(0,R);\mathbb{R}^N)$, where $R > 0$ is chosen as above. We now verify (a)–(c) from the Riesz-Kolmogorov compactness characterisation, Lemma 1.10.

Ad (a). Since $E\colon \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N) \to \mathrm{W}^{1,1}(\mathbb{R}^n;\mathbb{R}^N)$ is linear and bounded, we conclude

$$\begin{aligned}
\|\widetilde{v}_j\|_{\mathrm{W}^{1,1}(\mathbb{R}^n;\mathbb{R}^N)} &\leqslant C\|v_j\|_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \\
&\leqslant C\Big(\|u_j\|_{\mathrm{L}^1(\Omega;\mathbb{R}^N)} + |Du_j|(\Omega) + \frac{2}{j}\Big) \leqslant C < \infty
\end{aligned}$$

and therefore (a) is satisfied. Ad (b). We work from (a) and employ the standard estimate

$$\begin{aligned}
\int_{\mathbb{R}^n} |\widetilde{v}_j(x+h) - \widetilde{v}_j(x)|\,\mathrm{d}x &\leqslant C|h| \int_{\mathbb{R}^n} |D\widetilde{v}_j|\,\mathrm{d}x \\
&\leqslant C|h|\|v_j\|_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)} \leqslant C|h|
\end{aligned}$$

valid for all $h \in \mathbb{R}^n$. Thus, (b) is equally satisfied. Ad (c). This conditions holds trivially as every $\widetilde{v}_j$ has support contained in the fixed ball $\mathrm{B}(0,R)$.

Now, by Lemma 1.10, there exists a subsequence $(\widetilde{v}_{j(k)}) \subset (\widetilde{v}_j)$ and some $\widetilde{v} \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^N)$ such that $\widetilde{v}_{j(k)} \to \widetilde{v}$ strongly in $\mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^N)$.

Step 3. Conclusion. We put $u := \widetilde{v}|_\Omega$. We then record

$$\|u - v_{j(k)}\|_{\mathrm{L}^1(\Omega;\mathbb{R}^N)} = \|Eu - Ev_{j(k)}\|_{\mathrm{L}^1(\Omega;\mathbb{R}^N)} \leqslant \|\widetilde{v} - \widetilde{v}_{j(k)}\|_{\mathrm{W}^{1,1}(\mathbb{R}^n;\mathbb{R}^N)} \to 0$$

as $k \to \infty$. By the lower semicontinuity of the total variation and the estimate (2.37),

$$|Du|(\Omega) \leqslant \liminf_{k\to\infty} |Dv_{j(k)}|(\Omega) \leqslant C < \infty.$$

Hence, $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$. By (2.37), we moreover find

$$\|u-u_{j(k)}\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \leqslant \|u-v_{j(k)}\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} + \|v_{j(k)}-u_{j(k)}\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \to 0$$

as $k \to \infty$. It thus only remains to establish – for some further subsequence – $Du_{j(k(i))} \overset{*}{\rightharpoonup} Du$ in $\mathcal{M}(\Omega;\mathbb{R}^{N\times n})$. By assumption and the Banach-Alaoglu-Bourbaki theorem, there exists a subsequence $(u_{j(k(i))}) \subset (u_{j(k)})$ and some $\mu \in \mathcal{M}(\Omega;\mathbb{R}^{N\times n})$ such that $Du_{j(k(i))} \overset{*}{\rightharpoonup} \mu$ as $i \to \infty$ in $\mathcal{M}(\Omega;\mathbb{R}^{N\times n})$. We identify $\mu = Du$ as follows: Let $\varphi \in \mathrm{C}_c^\infty(\Omega;\mathbb{R}^{N\times n})$. Then we obtain

$$\begin{aligned} \langle \mu, \varphi\rangle &= \lim_{i\to\infty} \langle Du_{j(k(i))}, \varphi\rangle = -\lim_{i\to\infty}\langle u_{j(k(i))}, \mathrm{div}(\varphi)\rangle \\ &= -\langle u, \mathrm{div}(\varphi)\rangle = \langle Du, \varphi\rangle. \end{aligned}$$

Since φ was assumed arbitrary, $\mu = Du$. Now we redefine $u_{j(i)} := u_{j(k(i))}$, and the proof is complete. □

Equipped with these tools, we can now turn to the study of the specific properties of BV-maps. Here we assume the reader to familiar with the corresponding theory for $\mathrm{W}^{1,1}$-Sobolev maps as taught in introductory courses on functional analysis and/or PDEs. A good reference for the requisite background material is given by

- Brezis, H.: Functional Analyis, Sobolev Spaces and Partial Differential Equations. Springer Universitext, 2011.

We begin by extending the Sobolev inequality:

Theorem 2.38 (The Sobolev inequality)**.** *Let $n, N \geqslant 1$. Then there exists a constant $c = c(N,n) > 0$ such that for all $u \in \mathrm{BV}(\mathbb{R}^n;\mathbb{R}^N)$ there holds*

$$\|u\|_{\mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n;\mathbb{R}^N)} \leqslant c|Du|(\mathbb{R}^n). \tag{2.38}$$

Proof. For $u \in \mathrm{BV}(\mathbb{R}^n;\mathbb{R}^N)$ as in the theorem, we employ Theorem 2.36 to find $(u_j) \subset (\mathrm{BV}\cap\mathrm{C}^\infty)(\mathbb{R}^n;\mathbb{R}^N)$ such that $u_j \to u$ strictly in $\mathrm{BV}(\mathbb{R}^n;\mathbb{R}^N)$. Note that, by passing to a suitable (here non-relabeled) subsequence, we can achieve $u_j \to u$ pointwisely $\mathscr{L}^n$-a.e.. We then conclude by Fatou's lemma in the first step

$$\begin{aligned} \|u\|_{\mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n;\mathbb{R}^N)} &\leqslant \liminf_{j\to\infty} \|u_j\|_{\mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n;\mathbb{R}^N)} \\ &\leqslant c \liminf_{j\to\infty} |Du_j|(\mathbb{R}^n) \\ &\leqslant c|Du|(\mathbb{R}^n). \end{aligned}$$

Here we have used the Sobolev embedding $\mathrm{W}^{1,1}(\mathbb{R}^n;\mathbb{R}^N) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n;\mathbb{R}^N)$ in the penultimate step. The proof is complete. □

It is interesting to note that *in the context of* BV*-functions*, the Sobolev inequality can be interpreted geometrically. To this end, we make a brief digression. Let $\Omega \subset \mathbb{R}^n$ be measurable. We say that Ω *has finite perimeter* (in $\mathbb{R}^n$) provided

$$\mathbb{1}_\Omega \in \mathrm{BV}(\mathbb{R}^n),$$

and put $\mathrm{Per}(\Omega;\mathbb{R}^n) := |D\mathbb{1}_\Omega|(\mathbb{R}^n)$.

Sets of finite perimeter are sometimes also called *Caccioppoli sets*. To explain the particular terminology of *finite perimeter*, let us consider an open and bounded set $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. We aim to show that, in this situation,

$$\mathscr{H}^{n-1}(\partial\Omega) = \mathrm{Per}(\Omega;\mathbb{R}^n). \tag{2.39}$$

To this end, denote $\nu_{\partial\Omega}\colon \partial\Omega \to \mathbb{S}^{n-1}$ the outer unit normal to $\partial\Omega$ and employ Gauß' theorem to find for any $\varphi \in \mathrm{C}_c^1(\Omega;\mathbb{R}^n)$ with $|\varphi| \leqslant 1$

$$\int_{\mathbb{R}^n} \mathbb{1}_\Omega \operatorname{div}(\varphi)\,\mathrm{d}x = \int_\Omega \operatorname{div}(\varphi)\,\mathrm{d}x = \int_{\partial\Omega} \varphi\cdot\nu_{\partial\Omega}\,\mathrm{d}\mathscr{H}^{n-1}. \tag{2.40}$$

By passing to the maximum on the left hand side, we find by (2.31) $\mathrm{Per}(\Omega;\mathbb{R}^n) \leqslant \mathscr{H}^{n-1}(\partial\Omega)$. On the other hand, pick any[3] $\varphi \in \mathrm{C}_c^1(\mathbb{R}^n;\mathbb{R}^n)$ with $\varphi|_{\partial\Omega} = \nu_{\partial\Omega}$ and $|\varphi| \leqslant 1$. Then (2.40) conversely yields $\mathscr{H}^{n-1}(\partial\Omega) \leqslant \mathrm{Per}(\Omega;\mathbb{R}^n)$ and thus (2.39) follows.

Next, pick the optimal constant $c_{\mathrm{Sob}} > 0$ in the inequality of Theorem 2.38 with $N = 1$. If Ω is of finite perimeter, we consequently find

$$\mathscr{L}^n(\Omega)^{1-\frac{1}{n}} \leqslant c_{\mathrm{Sob}}\,\mathrm{Per}(\Omega;\mathbb{R}^n). \tag{2.41}$$

This inequality is usually referred to as *isoperimetric inequality.* To explain the importance of this inequality, let us note that a classical problem in geometry is to

$\boxed{\textit{find sets with maximal Lebesgue measure among all sets with fixed perimeter.}}$

____________________ April 18, 2019

It is a widely known (but in courses rarely proved) fact that balls do the job. Working from the isoperimetric inequality (2.41), one possible strategy to deal with this problem is to find the optimal constant $c_{\mathrm{Sob}} > 0$ in (2.41) and then to verify that balls in fact lead to equality in (2.41). In this sense, we shall establish

Proposition 2.39. *Let $n \in \mathbb{N}$. The optimal constant $c_{Sob} > 0$ in* (2.41) *is given by*

$$c_{Sob} = c_{Iso}.$$

The proof of Proposition 2.39 hinges on the so-called *coarea-formula* for BV-functions, which displays the next step in our program:

Theorem 2.40 (Coarea formula for BV-functions)**.** *Given $u \in \mathrm{BV}(\mathbb{R}^n)$, denote $E_t := \{x \in \mathbb{R}^n\colon\ u(x) > t\}$ the* superlevel set *of height t. Then the following hold:*

(a) $\mathrm{Per}(E_t;\mathbb{R}^n) < \infty$ *for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$,*
(b) *The total variation $|Du|(\mathbb{R}^n)$ can be represented as*

$$|Du|(\mathbb{R}^n) = \int_{-\infty}^{\infty} \mathrm{Per}(E_t;\mathbb{R}^n)\,\mathrm{d}t.$$

Conversely, if $u \in \mathrm{L}^1(\mathbb{R}^n)$ and

$$\int_{-\infty}^{\infty} \mathrm{Per}(E_t;\mathbb{R}^n)\,\mathrm{d}t < \infty,$$

then $u \in \mathrm{BV}(\mathbb{R}^n)$.

Proof. We divide the proof into two steps, discussing the upper and lower bounds each.

Step 1. The upper bound. Let $u \in (\mathrm{W}^{1,1} \cap \mathrm{C}^\infty)(\Omega)$. We put, for $t \in \mathbb{R}$,

$$m(t) := \int_{\{u \leqslant t\}} |Du|\,\mathrm{d}x.$$

[3] This can be established by several means; one option is an extension theorem due to Whitney.

Then m is non-decreasing and hence the derivative $m'(t)$ exists at $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$; see the remark below for an explanation. We then find

$$\int_{-\infty}^{\infty} m'(t)\,\mathrm{d}t \leqslant \int_{\mathbb{R}^n} |Du|\,\mathrm{d}x. \tag{2.42}$$

Let now $-\infty < t < \infty$ and the function $\rho\colon \mathbb{R} \to \mathbb{R}$ be given by

$$\rho(t) := \begin{cases} 0 & \text{if } s \leqslant t, \\ \frac{s-t}{r} & \text{if } t \leqslant s \leqslant t+r, \\ 1 & \text{if } s \geqslant t+r. \end{cases}$$

Then ρ is a Lipschitz function, and we have for all $\varphi \in \mathrm{C}^1_c(\Omega;\mathbb{R}^n)$ with $|\varphi| \leqslant 1$

$$\begin{aligned} -\int_\Omega \rho(u(x))\,\mathrm{div}(\varphi)\,\mathrm{d}x &= \int_\Omega \rho'(u(x))Du(x)\cdot\varphi\,\mathrm{d}x \\ &= \frac{1}{r}\int_{\{t\leqslant|u|\leqslant t+r\}} Du\cdot\varphi\,\mathrm{d}x = \frac{m(t+r)-m(t)}{r}. \end{aligned}$$

Now suppose that t is such that m is differentiable at t; by the above, $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$ does this job. Then we have by passing $r \to 0$

$$-\int_{E_t} \mathrm{div}(\varphi)\,\mathrm{d}x \leqslant m'(t),$$

and passing to the supremum over all φ as specified above,

$$\mathrm{Per}(E_t;\Omega) \leqslant m'(t).$$

Integrating the previous inequality with respect to $t \in \mathbb{R}$ then yields by (2.42)

$$\int_{-\infty}^{\infty} \mathrm{Per}(E_t;\Omega)\,\mathrm{d}t \leqslant \int_{\mathbb{R}^n} |Du|\,\mathrm{d}x. \tag{2.43}$$

To extend this estimate to general functions $u \in \mathrm{BV}(\Omega)$, we employ Theorem 2.36 to find a sequence $(u_j) \subset (\mathrm{C}^\infty \cap \mathrm{BV})(\Omega)$ such that $u_j \to u$ strictly in $\mathrm{BV}(\Omega)$. We define $E_t^j := \{x \in \Omega\colon\; u_j(x) > t\}$. Then there holds

$$\int_{-\infty}^{\infty} |\mathbb{1}_{E_t^j}(x) - \mathbb{1}_{E_t}(x)|\,\mathrm{d}t = |u_j(x) - u(x)| \tag{2.44}$$

for $\mathscr{L}^n$-a.e. $x \in \Omega$, where $u(x)$ is understood as the Lebesgue value of u at x. Using (2.44), we obtain by Fubini

$$\begin{aligned} \int_{-\infty}^{\infty}\int_\Omega |\mathbb{1}_{E_t^j}(x) - \mathbb{1}_{E_t}(x)|\,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega\int_{-\infty}^{\infty}\int_\Omega |\mathbb{1}_{E_t^j}(x) - \mathbb{1}_{E_t}(x)|\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_\Omega |u_j(x) - u(x)|\,\mathrm{d}x \to 0, \end{aligned}$$

and so we may pass to subsequence $(u_{j(i)}) \subset (u_j)$ to achieve

$$\int_\Omega |\mathbb{1}_{E_t^{j(i)}}(x) - \mathbb{1}_{E_t}(x)|\,\mathrm{d}x \to 0$$

for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$. Thus, for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$, $\mathbb{1}_{E_t^{j(i)}} \to \mathbb{1}_{E_t}$ in $\mathrm{L}^1(\Omega)$ and hence, by Lemma 2.35,

$$\mathrm{Per}(E_t;\Omega) = |D\mathbb{1}_{E_t}|(\Omega) \leqslant \liminf_{i\to\infty} |D\mathbb{1}_{E_t^{j(i)}}|(\Omega).$$

We then integrate the preceding inequality with respect to $t \in \mathbb{R}$ and employ Fatou's lemma to obtain by (2.43)

$$\int_{-\infty}^{\infty} \mathrm{Per}(E_t;\Omega)\,\mathrm{d}t \leqslant \liminf_{i\to\infty} \int_{-\infty}^{\infty} |D\mathbb{1}_{E_t^{j(i)}}|(\Omega)\,\mathrm{d}t \overset{(2.43)}{\leqslant} \int_{\mathbb{R}^n} |Du_{j(i)}|\,\mathrm{d}x = |Du|(\Omega),$$

the last equality being valid by $u_j \to u$ strictly in $\mathrm{BV}(\Omega)$ as $j \to \infty$.

Step 2. The lower bound. For the lower bound, we let $u \in \mathrm{L}^1(\Omega)$. Then, if $u \geqslant 0$, we obtain for $\mathscr{L}^n$-a.e. $x \in \Omega$

$$u(x) = \int_0^{u(x)} \mathrm{d}t = \int_0^{\infty} \mathbb{1}_{E_t}(x)\,\mathrm{d}t$$

and hence, letting $\varphi \in \mathrm{C}_c^1(\Omega;\mathbb{R}^n)$ satisfy $|\varphi| \leqslant 1$, we obtain

$$\int_{\Omega} u\,\mathrm{div}(\varphi)\,\mathrm{d}x = \int_0^{\infty}\int_{\Omega} \mathbb{1}_{E_t}\,\mathrm{div}(\varphi)\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^{\infty} \mathrm{Per}(E_t;\Omega)\,\mathrm{d}t.$$

Now pass to the supremum to conclude in the case where $u \geqslant 0$. If $u \leqslant 0$, we similarly employ

$$u(x) = \int_{-\infty}^{0} (1 - \mathbb{1}_{E_t}(x))\,\mathrm{d}t$$

and argue similarly. In the general case, split $u = u^+ - u^-$, $u^{\pm}$ denoting the positive or negative parts of u, respectively, apply the previous inequalities to the single parts and put the estimates together. The proof is complete. $\square$

To approach Proposition 2.39, we start by giving a *lower bound* on c_{Sob}. Clearly, $\mathbb{1}_{\mathrm{B}(0,1)} \in \mathrm{BV}(\mathbb{R}^n)$, and we have by Theorem 2.38 and the discussion after (2.40)

$$\omega^{\frac{n-1}{n}} \leqslant c_{\mathrm{Sob}} \mathscr{H}^{n-1}(\partial \mathrm{B}(0,1)) = c_{\mathrm{Sob}} n\omega_n.$$

so that $1/(n\sqrt[n]{\omega_n}) \leqslant c_{\mathrm{Sob}}$. Suppose now that

$$c_{\mathrm{Iso}} \leqslant \frac{1}{n\sqrt[n]{\omega_n}}, \tag{2.45}$$

an inequality that we shall establish below. Then the following argument shows that $c_{\mathrm{Sob}} \leqslant c_{\mathrm{Iso}}$ and so we shall have established Proposition 2.39.

To see $c_{\mathrm{Sob}} \leqslant c_{\mathrm{Iso}}$, let $u \in (\mathrm{BV} \cap \mathrm{C}^{\infty})(\mathbb{R}^n)$. We note in advance that, because $t \mapsto t\mathscr{L}^n(\{|u| \geqslant t\})^{\frac{n}{n-1}}$ is non-increasing,

$$t\mathscr{L}^n(\{|u| \geqslant t\})^{\frac{n-1}{n}} \leqslant \int_0^t \tau \mathscr{L}^n(\{|u| \geqslant \tau\})^{\frac{n}{n-1}}\,\mathrm{d}\tau$$

and thus, raising the preceding inequality to the power $\frac{1}{n-1}$,

$$t^{\frac{1}{n-1}} \mathscr{L}^n(\{|u| \geqslant t\})^{\frac{1}{n}} \leqslant \Big(\int_0^t \mathscr{L}^n(\{|u| \geqslant \tau\})^{\frac{n-1}{n}}\,\mathrm{d}\tau\Big)^{\frac{1}{n-1}}. \tag{2.46}$$

Let us moreover define

$$\Phi(t) := \Big(\int_0^t \mathscr{L}^n(\{|u| \geqslant \tau\})^{\frac{n-1}{n}}\,\mathrm{d}\tau\Big)^{\frac{n}{n-1}}.$$

We then compute the derivative of Φ as

$$\Phi'(t) = \frac{n}{n-1}\mathscr{L}^n(\{|u| \geqslant t\})^{\frac{n-1}{n}}\Big(\int_0^t \mathscr{L}^n(\{|u| \geqslant \tau\})^{\frac{n-1}{n}}\,\mathrm{d}\tau\Big)^{\frac{1}{n-1}}.$$

Therefore,

$$
\begin{aligned}
\|u\|_{\mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n)} &= \Big(\int_0^\infty \frac{n}{n-1} t^{\frac{1}{n-1}} \mathscr{L}^n(\{|u| \geqslant t\})\,\mathrm{d}t\Big)^{\frac{n-1}{n}} \\
&= \Big(\int_0^\infty \frac{n}{n-1}\Big(t^{\frac{1}{n-1}}\mathscr{L}^n(\{|u| \geqslant t\})^{\frac{1}{n}}\Big)\mathscr{L}^n(\{|u| \geqslant t\})^{\frac{n-1}{n}}\,\mathrm{d}t\Big)^{\frac{n-1}{n}} \\
&\overset{(2.46)}{\leqslant} \Big(\int_0^\infty \frac{n}{n-1}\Big(\int_0^t \mathscr{L}^n(\{|u| \geqslant \tau\})^{\frac{n-1}{n}}\,\mathrm{d}\tau\Big)^{\frac{1}{n-1}}\mathscr{L}^n(\{|u| \geqslant t\})^{\frac{n-1}{n}}\,\mathrm{d}t\Big)^{\frac{n-1}{n}} \\
&= \Big(\int_0^\infty \Phi'(t)\,\mathrm{d}t\Big)^{\frac{n-1}{n}} = \lim_{t\to\infty}\Phi(t)^{\frac{n-1}{n}} = \int_0^\infty \mathscr{L}^n(\{|u| \geqslant t\})^{\frac{n-1}{n}}\,\mathrm{d}t,
\end{aligned}
$$

and by definition of Φ, we thus obtain

$$
\begin{aligned}
\|u\|_{\mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n)} &\leqslant \int_0^\infty \mathscr{L}^n(\{|u| \geqslant t\})^{\frac{n-1}{n}}\,\mathrm{d}t \\
&\leqslant c_{\mathrm{Iso}}\int_0^\infty \mathrm{Per}(\{|u| \geqslant t\};\mathbb{R}^n)\,\mathrm{d}t \\
&= c_{\mathrm{Iso}}\int_0^\infty \mathrm{Per}(\{u \geqslant t\};\mathbb{R}^n)\,\mathrm{d}t + c_{\mathrm{Iso}}\int_0^\infty \mathrm{Per}(\{u \leqslant -t\};\mathbb{R}^n)\,\mathrm{d}t \\
&= c_{\mathrm{Iso}}\int_{-\infty}^\infty |Du|\,\mathrm{d}x,
\end{aligned}
$$

and hence $c_{\mathrm{Iso}} \geqslant c_{\mathrm{Sob}}$. But we have already established that $c_{\mathrm{Iso}} \leqslant c_{\mathrm{Sob}}$, and hence we must have equality.

Thus, the only estimate that remains to be proved is given by (2.45), and we approach this estimate via the following geometric inequality:

Lemma 2.41 (Brunn-Minkowski). *Let $A, B \subset \mathbb{R}^n$ be two compact sets. Then there holds*

$$
\mathscr{L}^n(A)^{\frac{1}{n}} + \mathscr{L}^n(B)^{\frac{1}{n}} \leqslant \mathscr{L}^n(A+B)^{\frac{1}{n}}, \tag{2.47}
$$

where $A + B := \{x + y\colon\ x \in A,\ y \in B\}$ is the sum set of A and B as usual.

The Brunn-Minkowski lemma can be approached via the Prékopa-Leindler inequality, which we state here as an auxiliary tool:

Lemma 2.42. *Let $0 < \theta < 1$ and let $f, g, h\colon \mathbb{R}^m \to \mathbb{R}$ be non-negative, measurable functions that satisfy*

$$
f(x)^{1-\theta}g(x)^{\theta} \leqslant h(x+y) \qquad \text{for all } x, y \in \mathbb{R}^d.
$$

Then there holds

$$
\Big(\int_{\mathbb{R}^m} f\,\mathrm{d}x\Big)^{1-\theta}\Big(\int_{\mathbb{R}^m} g\,\mathrm{d}x\Big)^{\theta} \leqslant (1-\theta)^{m(1-\theta)}\theta^{m\theta}\int_{\mathbb{R}^d} h\,\mathrm{d}x.
$$

We can now proceed to showing (2.45). To this end, let $A \subset \mathbb{R}^n$ be Lebesgue measurable and let $\varepsilon > 0$. Then we find

$$
\Big(\mathscr{L}^n(A)^{\frac{1}{n}} + \omega_n^{\frac{1}{n}}\varepsilon\Big) \leqslant \mathscr{L}^n(A + \overline{\mathrm{B}}(0,\varepsilon))^{\frac{1}{n}}.
$$

We raise this inequality to the power n, shift $\mathscr{L}^n(A)$ to the right-hand side and divide by ε, yielding

$$\sum_{k=1}^{n}\binom{n}{k}\omega_n^{\frac{k}{n}}\varepsilon^{k-1}\mathscr{L}^n(A)^{\frac{n-k}{n}} \leqslant \frac{\mathscr{L}^n(A+\overline{\mathrm{B}}(0,\varepsilon))-\mathscr{L}^n(A)}{\varepsilon}.$$

Sending $\varepsilon \searrow 0$, we thereby find

$$n\omega_n^{\frac{1}{n}}\mathscr{L}^n(A)^{\frac{n-1}{n}} \leqslant \lim_{\varepsilon\searrow 0}\frac{\mathscr{L}^n(A+\overline{\mathrm{B}}(0,\varepsilon))-\mathscr{L}^n(A)}{\varepsilon},$$

and we need to argue that

$$\lim_{\varepsilon\searrow 0}\frac{\mathscr{L}^n((A+\overline{\mathrm{B}}(0,\varepsilon))\setminus A)}{\varepsilon} \leqslant \mathscr{H}^{n-1}(A).$$

This I will insert in due course.

———————————————————————— April 25, 2019

A digression on potential theory

We now take this opportunity to dig a bit deeper and aim to clarify the underlying mechanisms for the Sobolev inequality to hold. The usual approach to the Sobolev inequality is given by *slicing* – to be recalled below – and we here aim to give a more potential theoretical interpretation. Consider for $f \in \mathrm{L}^p(\mathbb{R}^n)$ the equation

$$-\Delta u = f. \tag{2.48}$$

Without appealing to hard tools, let us inquire what can be said about the integrability of the gradients. In this respect, we require the following

Theorem 2.43 (Fractional Integration Theorem)**.** *Let $0 < s < n$, $1 < p < \infty$ such that $sp < n$. Define the* Riesz potential operator *I_s of order s by*

$$I_s f(x) := \int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-s}}\,\mathrm{d}y, \qquad x \in \mathbb{R}^n. \tag{2.49}$$

Then I_s maps $\mathrm{L}^p(\mathbb{R}^n) \to \mathrm{L}^{\frac{np}{n-sp}}(\mathbb{R}^n)$ boundedly. If $p = 1$, $I_s\colon\ \mathrm{L}^1(\mathbb{R}^n) \to \mathrm{L}_w^{\frac{n}{n-s}}(\mathbb{R}^n)$.

Note that the preceding theorem cannot be extended to $p = 1$. To prove Theorem 2.43, we need some more machinery and start by recalling the *Hardy-Littlewood maximal operator $\mathcal{M}$*. For $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ and $x \in \mathbb{R}^n$, we put

$$\mathcal{M}f(x) := \sup_{r>0}⨍_{\mathrm{B}(x,r)}|f|\,\mathrm{d}y.$$

The following theorem is standard, and we record it here for completeness:

Proposition 2.44 (Boundedness of $\mathcal{M}$)**.** *There exists a constant $c > 0$ such that for all $f \in \mathrm{L}^1(\mathbb{R}^n)$ there holds*

$$\sup_{t>0} t\mathscr{L}^n(\{x\colon \mathbb{R}^n\colon\ \mathcal{M}f(x) > t\}) \leqslant c\|f\|_{\mathrm{L}^1(\mathbb{R}^n)}. \tag{2.50}$$

Moreover, for each $1 < p \leqslant \infty$ there exists $c_p > 0$ such that for all $f \in \mathrm{L}^p(\mathbb{R}^n)$ there holds

$$\|\mathcal{M}f\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant c_p\|f\|_{\mathrm{L}^p(\mathbb{R}^n)}. \tag{2.51}$$

Let us now come to the

Proof of 2.43. Let $f \in \mathrm{L}^p(\mathbb{R}^n)$, with $1 < p < \infty$ such that $sp < n$, and let $x \in \mathbb{R}^n$ be arbitrary but fixed. The idea of the proof is to split the domain of integration in (2.49) into a part consisting of elements *close* to x and a part consisting of elements *far* from x. The former shall be dealt with the Hardy-Littlewood-Wiener theorem, Theorem 2.44, and the latter is tackled with Hölder's inequality. To this end, let $r > 0$ be arbitrary, and split

$$I_s f(x) = \int_{\mathrm{B}(x,r)} \frac{f(y)}{|x-y|^{n-s}}\,\mathrm{d}y + \int_{\mathbb{R}^n \setminus \mathrm{B}(x,r)} \frac{f(y)}{|x-y|^{n-s}}\,\mathrm{d}y =: \mathbf{I} + \mathbf{II}.$$

Ad $\mathbf{I}$. We split the domain of integration $\mathrm{B}(x,r)$ into a countable family of annuli via

$$\mathrm{B}(x,r) = \bigcup_{k=0}^{\infty} \mathcal{A}_k := \bigcup_{k=0}^{\infty} \Big(\mathrm{B}(x, \frac{r}{2^k}) \setminus \mathrm{B}(x, \frac{r}{2^{k+1}}) \Big).$$

Note that if $x \in \mathcal{A}_k$, then $|x-y| > \frac{r}{2^{k+1}}$ and thus $|x-y|^{-n+s} < r^{-n+s} 2^{(k+1)(n-s)}$. Thus,

$$\begin{aligned} |\mathbf{I}| &\leqslant \sum_{k=0}^{\infty} \int_{\mathcal{A}_k} \frac{|f(y)|}{|x-y|^{n-s}}\,\mathrm{d}y \\ &\leqslant \sum_{k=0}^{\infty} \Big(\frac{r}{2^{k+1}}\Big)^{s-n} \int_{\mathcal{A}_k} |f(y)|\,\mathrm{d}y \leqslant \sum_{k=0}^{\infty} \Big(\frac{r}{2^{k+1}}\Big)^{s-n} \int_{\mathrm{B}(x,\frac{r}{2^k})} |f(y)|\,\mathrm{d}y = (*), \end{aligned}$$

and the last inequality holds because of $\mathcal{A}_k \subset \mathrm{B}(x, \frac{r}{2^k})$. To employ the Hardy-Littlewood maximal operator as indicated above, we rewrite and estimate as follows:

$$\begin{aligned} \mathbf{I} \leqslant (*) &\leqslant \omega_n \sum_{k=0}^{\infty} \Big(\frac{r}{2^{k+1}}\Big)^{s-n} \Big(\frac{r}{2^k}\Big)^{n} \fint_{\mathrm{B}(x,\frac{r}{2^k})} |f(y)|\,\mathrm{d}y \\ &\leqslant \omega_n (\mathcal{M}f(x)) \frac{r^s}{2^{s-n}} \Big(\sum_{k=0}^{\infty} \frac{1}{2^{sk}} \Big) \qquad \Big(\text{as} \fint_{\mathrm{B}(x,\frac{r}{2^k})} |f(y)|\,\mathrm{d}y \leqslant \mathcal{M}f(x)\Big) \\ &\leqslant c(s,n,p) r^s (\mathcal{M}f(x)), \end{aligned}$$

since due to $s > 0$, the geometric series occuring in the penultimate step converges. Ad $\mathbf{II}$. As explained above, we use Hölder's inequality in the first and employ polar coordinates in the second step to arrive at

$$\begin{aligned} \mathbf{II} &\leqslant \Big(\int_{\mathbb{R}^n \setminus \mathrm{B}(x,r)} |f(y)|^p\,\mathrm{d}y \Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}^n \setminus \mathrm{B}(x,r)} \frac{\mathrm{d}y}{|x-y|^{(n-s)p'}} \Big)^{\frac{1}{p'}} \\ &\leqslant (n\omega_n)^{\frac{1}{p'}} \|f\|_{\mathrm{L}^p(\mathbb{R}^n)} \Big(\int_r^{\infty} t^{n-1-(n-s)p'}\,\mathrm{d}t \Big)^{\frac{1}{p'}} = (**). \end{aligned}$$

Now note that our assumption $sp < n$ implies that $n - 1 - (n-s)p' < -1$ and hence the second integral exists. More precisely,

$$(**) \leqslant \Big(\frac{n\omega_n}{(n-(n-s)p')} \Big)^{\frac{1}{p'}} r^{s-\frac{n}{p}} \|f\|_{\mathrm{L}^p(\mathbb{R}^n)} = c(s,n,p) r^{s-\frac{n}{p}} \|f\|_{\mathrm{L}^p(\mathbb{R}^n)}.$$

Gathering the estimates for $\mathbf{I}$ and $\mathbf{II}$, we thus obtain

$$|I_s f(x)| \leqslant c(s,n,p) \Big(r^s (\mathcal{M}f(x)) + r^{s-\frac{n}{p}} \|f\|_{\mathrm{L}^p(\mathbb{R}^n)} \Big), \tag{2.52}$$

and at this stage we recall that we still have the freedom to choose $r > 0$ which might indeed depend on x. So we set $r := r_x := (\mathcal{M}f(x)/\|f\|_{\mathrm{L}^p(\mathbb{R}^n)})^{-p/n}$. With this choice of r, (2.52) becomes

$$|I_s f(x)| \leqslant c(s,n,p)\|f\|_{\mathrm{L}^p(\mathbb{R}^n)}^{\frac{sp}{n}}(\mathcal{M}f(x))^{1-\frac{sp}{n}}.$$

Now we raise both sides of the previous inequality to the power $\frac{np}{n-sp}$ and integrate over $\mathbb{R}^n$ to obtain

$$\|I_s f\|_{\mathrm{L}^{\frac{np}{n-sp}}(\mathbb{R}^n)}^{\frac{np}{n-sp}} \leqslant c(s,n,p)\|f\|_{\mathrm{L}^p(\mathbb{R}^n)}^{\frac{sp}{n}\frac{np}{n-sp}}\int_{\mathbb{R}^n}|\mathcal{M}f(x)|^p\,\mathrm{d}x \leqslant c(s,n,p)\|f\|_{\mathrm{L}^p(\mathbb{R}^n)}^{p+\frac{sp}{n}\frac{np}{n-sp}}.$$

Here we have used the L^p-boundedness of the Hardy-Littlewood maximal operator (recall that $p > 1$ by assumption). Now it suffices to realise that

$$p + \frac{sp}{n}\frac{np}{n-sp} = \frac{np}{n-sp},$$

and so taking the $\frac{np}{n-sp}$-th root on both sides of the last inequality yields the claim. The proof is complete. $\square$

The fractional integration theorem immediately implies the Sobolev embedding theorem in $\mathrm{W}^{1,1}(\mathbb{R}^n)$ – and thereby, employing an approximation procedure as done in the proof of Theorem 2.38 – in $\mathrm{BV}(\mathbb{R}^n)$. To see how, let $u \in \mathrm{C}_c^\infty(\mathbb{R}^n)$ and recall that we can represent any $u \in \mathrm{C}_c^\infty(\mathbb{R}^n)$ via

$$u(x) = \frac{1}{n\omega_n}\int_{\mathbb{R}^n}\frac{\nabla u(y)\cdot(x-y)}{|x-y|^n}\,\mathrm{d}y, \qquad x \in \mathbb{R}^n.$$

Suppose that $Du \in \mathrm{L}^p(\mathbb{R}^n;\mathbb{R}^n)$ for some $1 < p < n$. Then we obtain

$$|u(x)| \leqslant \frac{1}{n\omega_n}|I_1(Du)(x)|,$$

and by Theorem 2.43, $I_1(Du) \in \mathrm{L}^{\frac{np}{n-p}}(\mathbb{R}^n)$ with $\|I_1(Du)\|_{\mathrm{L}^{\frac{np}{n-p}}(\mathbb{R}^n)} \leqslant C\|Du\|_{\mathrm{L}^p(\mathbb{R}^n)}$. We consequently obtain

$$\|u\|_{\mathrm{L}^{\frac{np}{n-p}}(\mathbb{R}^n)} \leqslant C\|I_1(Du)\|_{\mathrm{L}^{\frac{np}{n-p}}(\mathbb{R}^n)} \leqslant C\|Du\|_{\mathrm{L}^p(\mathbb{R}^n)},$$

and this is exactly the Sobolev inequality. However, it does not immediately apply to the limiting case $p = 1$, and for this we need an improvement due to Maz'ya as follows.

Lemma 2.45 (Maz'ya truncation)**.** *Let (X,Σ) be a measurable space and let μ,ν be two positive measures on Σ. Given $1 \leqslant p \leqslant q < \infty$, suppose that $u \in \mathrm{L}_w^q(X;\mu)$ and $g \in \mathrm{L}^p(X;\nu)$. Moreover, assume that $(A_t)_{t>0}$ is a decreasing family of measurable subsets of X in the sense of set inclusion. If for every $0 < t_1 < t_2 < \infty$ there holds*

$$\sup_{t>0} t^q\mu(\{u_{t_1}^{t_2} > t\}) \leqslant \|g\mathbb{1}_{A_{t_1}\setminus A_{t_2}}\|_{\mathrm{L}^p(X;\nu)}^q, \tag{2.53}$$

then we have $u \in \mathrm{L}^q(X,\nu)$ and there holds

$$\|u\|_{\mathrm{L}^q(X;\mu)} \leqslant 4\|g\|_{\mathrm{L}^p(X;\nu)}. \tag{2.54}$$

Here, the function $u_{t_1}^{t_2}\colon \mathbb{R}^n \to \mathbb{R}$ *is defined by* $u_{t_1}^{t_2} := \widetilde{u}_{t_1}^{t_2} - t_1$*, where*

$$\widetilde{u}_{t_1}^{t_2} := \begin{cases} t_2 & \text{if } |u(x)| \geqslant t_2, \\ |u(x)| & \text{if } t_1 \leqslant |u(x)| \leqslant t_2, \\ t_1 & \text{if } |u(x)| \leqslant t_1. \end{cases}$$

One occasionally refers to $u_{t_1}^{t_2}$ as the *Maz'ya truncation at heights* t_1, t_2.

Proof. We note in advance that, for all $k \in \mathbb{Z}$,

$$u_{2^{k-2}}^{2^{k-1}} \geqslant 2^{k-2} \Leftrightarrow \widetilde{u}_{2^{k-2}}^{2^{k-1}} \geqslant 2^{k-1} \Leftrightarrow |u(x)| \geqslant 2^{k-1}.$$

Hence,

$$\begin{aligned} \int_X |u|^q \,\mathrm{d}\mu &\leqslant \sum_{k\in\mathbb{Z}} 2^{kq} \mu(\{2^{k-1} \leqslant |u|\}) \\ &\leqslant \sum_{k\in\mathbb{Z}} 2^{kq} \mu(\{u_{2^{k-2}}^{2^{k-1}} \geqslant 2^{k-2}\}) \\ &\overset{(2.53)}{\leqslant} \sum_{k\in\mathbb{Z}} 2^{kq} 2^{-(k-2)q} \Big(\int_{A_{2^{k-2}} \setminus A_{2^{k-1}}} |g|^p \,\mathrm{d}\nu \Big)^{\frac{q}{p}} \\ &\leqslant 2^{2q} \Big(\sum_{k\in\mathbb{Z}} \int_{A_{2^{k-2}} \setminus A_{2^{k-1}}} |g|^p \,\mathrm{d}\nu \Big)^{\frac{q}{p}} \\ &\leqslant 2^{2q} \|g\|_{\mathrm{L}^p(X;\nu)}^q. \end{aligned}$$

Here we have used that $\ell^1(\mathbb{Z}) \hookrightarrow \ell^s(\mathbb{Z})$ for any $1 \leqslant s \leqslant \infty$. The proof is complete. □

As a consequence, since $|Du_{t_1}^{t_2}| = |Du| \mathbb{1}_{\{t_1 < |u| \leqslant t_2\}}$ for $u \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, we thus obtain *a proof of the Sobolev inequality for* $p = 1$*, combining the fractional integration theorem and the Maz'ya truncation lemma.*

Let us note that the fractional integration does not yield boundedness of the Riesz potential operators I_1 from $\mathrm{L}^1(\mathbb{R}^n)$ to $\mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n)$; to obtain a correct result, we have to replace the latter space by the weak Lebesgue space $\mathrm{L}_w^{\frac{n}{n-1}}(\mathbb{R}^n)$. Indeed, it might happen that $f \in \mathrm{L}^1$, $\Delta u = f$, but $Du \notin \mathrm{L}_{\mathrm{loc}}^{\frac{n}{n-1}}$. It thus stands to reason to inquire what can be said in this limiting case *at all.* In fact, the non-availability of the strong $(1, \frac{n}{n-1})$-estimate in the fractional integration theorem is not the end of the story: Sometimes something survives, so for instance here:

Example 2.46 (Brezis & Van Schaftingen, 2008)**.** *Let* $\Omega \subset \mathbb{R}^2$ *be open, bounded and connected and suppose that* $f \in \mathrm{L}^1(\Omega; \mathbb{R}^2)$ *is distributionally solenoidal (i.e., divergence-free) in the sense that*

$$\int_\Omega \langle f, \nabla\varphi \rangle \,\mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{C}_c^1(\Omega; \mathbb{R}^2).$$

Then there exists a constant $c > 0$ *independent of* f *such that every weak solution* u *of the equation*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

satisfies

$$\|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^2)} + \|u\|_{\mathrm{W}^{1,2}(\Omega;\mathbb{R}^2)} \leqslant c\,\|f\|_{\mathrm{L}^1(\Omega;\mathbb{R}^2)}. \tag{2.55}$$

Note that the existence of $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^2)$*-solution is not clear at all here. What is actually clear by virtue of the fundamental solution representation of* u *via* f *is that there exists* $u \in \mathrm{W}^{1,q}(\Omega;\mathbb{R}^2)$ *for every* $1 \leqslant q < 2$*, but* $q = 2$ *does not follow by straightforward potential theoretic methods. Now,* even if we can show that $q = 2$ is allowed per se, *in two dimensions we have* $\mathrm{W}^{1,2} \not\hookrightarrow \mathrm{L}^\infty$.

To approach the $\mathrm{W}^{1,2}$*-bound, we deduce from simple connectedness of* Ω *and* $\mathrm{div}(f) = 0$ *in* $\mathscr{D}'(\Omega)$ *that there exists* $F \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^2)$ *such that* $f = (\partial_2 F, -\partial_1 F)$*. Then we estimate, writing* $\varphi = (\varphi_1, \varphi_2)$

$$\begin{aligned}\int_\Omega \langle Du, D\varphi\rangle\,\mathrm{d}x &= \int_\Omega \langle -\Delta u, \varphi\rangle\,\mathrm{d}x = \int_\Omega \langle f, \varphi\rangle\,\mathrm{d}x \\ &= \int_\Omega |F|\,|D\varphi|\,\mathrm{d}x \leqslant \|F\|_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^2)}\|\varphi\|_{\mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)}.\end{aligned}$$

having used $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^2) \hookrightarrow \mathrm{L}^2(\Omega;\mathbb{R}^2)$ *in two dimensions*

Example 2.47 (Example 2.46 in higher dimensions)**.** *If we aim to generalise the previous example to arbitrary dimensions, note that if* $n \geqslant 3$*, then not every* $f \in \mathrm{L}^1(\Omega;\mathbb{R}^n)$ *with* $\mathrm{div}(f) = 0$ *can be written as a differential of some* $F \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^n)$*. We only sketch the general idea, using a deep tool due to* Bourgain & Brezis*. Recall that if* $n \geqslant 2$*, then*

$$\mathrm{W}^{1,n}(\mathbb{R}^n) \hookrightarrow \mathrm{BMO}(\mathbb{R}^n), \quad \mathrm{W}^{1,n}(\mathbb{R}^n) \not\hookrightarrow \mathrm{L}^\infty(\mathbb{R}^n). \tag{2.56}$$

Bourgain & Brezis *established the following decomposition result: Every* $u \in \mathrm{W}^{1,n}(\mathbb{R}^n;\mathbb{R}^n)$ *can be decomposed into* $u = \psi + \nabla\varphi$*, where*

(a) $\psi \in \mathrm{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)$ *with* $\|\psi\|_{\mathrm{L}^\infty} \leqslant c\|u\|_{\mathrm{W}^{1,n}}$.
(b) $\varphi \in \mathrm{W}^{2,n}(\mathbb{R}^n)$ *with* $\|\varphi\|_{\mathrm{W}^{2,n}} \leqslant c\|u\|_{\mathrm{W}^{1,n}}$.

Informally, the failure of the embedding of $\mathrm{W}^{1,n}$ *into* L^∞ *in* (2.56) *is due to the presence of gradient terms in the Bourgain-Brezis decomposition of* u*. We shall not prove this result, but rather use it. Similar as above, for all* $\varphi \in \mathrm{W}_0^{1,n}(\Omega)$

$$\int_\Omega \langle Du, D\varphi\rangle\,\mathrm{d}x = \int_\Omega \langle f, \eta + \nabla\psi\rangle\,\mathrm{d}x = \int_\Omega \langle f, \eta\rangle\,\mathrm{d}x \leqslant \|f\|_{\mathrm{L}^1}\|\varphi\|_{\mathrm{W}^{1,n}},$$

and at this stage we may pass to the supremum over all $\varphi \in \mathrm{W}_0^{1,n}(\Omega)$ *with* $\|\varphi\|_{\mathrm{W}^{1,n}} \leqslant 1$ *to conclude the* $\mathrm{L}^{\frac{n}{n-1}}$*-bound on* Du.

Back to the main theme: BV functions

Theorem 2.48 (Trace theorem)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded with Lipschitz boundary* $\partial\Omega$*. Then there exists a linear trace operator* $\mathrm{Tr}_{\partial\Omega}\colon \mathrm{BV}(\Omega;\mathbb{R}^N) \to \mathrm{L}^1(\partial\Omega;\mathbb{R}^N)$*. This operator is continuous for the strict topology.*

Proof. TBI. □

——————————————————————— April 29, 2019

Theorem 2.49 (Extensions and Gluing)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be an open and bounded Lipschitz set.*

(a) *There exists a bounded*[4] *linear extension operator* $E\colon \mathrm{W}^{1,1}(\Omega) \to \mathrm{W}^{1,1}(\mathbb{R}^n)$.

[4]With respect to the norm topology on $\mathrm{W}^{1,1}$.

(b) *If* $v \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ *and* $w \in \mathrm{BV}(\overline{\Omega}^c;\mathbb{R}^N)$*, then the* glued map $u\colon \mathbb{R}^n \to \mathbb{R}^N$ *which is defined for* $\mathscr{L}^n$*–a.e.* $x \in \mathbb{R}^n$ *by*

$$u(x) := \begin{cases} v(x) & \text{for } x \in \Omega, \\ w(x) & \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega} \end{cases}$$

belongs to $\mathrm{BV}(\mathbb{R}^n;\mathbb{R}^N)$*, and its gradient* $Du \in \mathscr{M}(\mathbb{R}^n;\mathbb{R}^{N\times n})$ *is given by*

$$Du = Dv \llcorner \Omega + Dw \llcorner \overline{\Omega}^c + (\mathrm{Tr}(v) - \mathrm{Tr}(w)) \otimes \nu_{\partial\Omega}\mathscr{H}^{n-1} \llcorner \partial\Omega, \tag{2.57}$$

where $\nu_{\partial\Omega}$ *denotes the outward unit normal to* $\partial\Omega$ *and* $\mathrm{Tr}(v), \mathrm{Tr}(w)$ *the outer traces of* v *and* w *on* $\partial\Omega$*, respectively.*

2.5. **Convex functionals of measures and existence of generalised minima.** In this section we study a canonical way of how to apply convex functions to measures. To this end let $f\colon \Omega \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ and let $\mu \in \mathscr{M}(\Omega;\mathbb{R}^m)$ be an $\mathbb{R}^m$-valued Radon measure with finite total variation. Before we delve into the solid framework, we begin with some heuristics helping us to get an idea of what to expect.

For simplicity, let us assume that f has no explicit x-dependence. As usual, we denote $|\mu|^s$ the singular part of $|\mu|$ with respect to $\mathscr{L}^n$. Approximating μ by a sequence (μ_ε) of measures which are absolutely continuous with respect to $\mathscr{L}^n$ and satisfy $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ as $\varepsilon \searrow 0$, we pick $A \in \mathscr{B}(\Omega)$ such that $\mathscr{L}^n(A) = |\mu^s|(\Omega \setminus A) = 0$. For $\mathscr{L}^n$-a.e. $x \in A$, we observe that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}(x) = \lim_{\varepsilon \searrow 0} \frac{\mu(\mathrm{B}(x,\varepsilon))}{\mathscr{L}^n(\mathrm{B}(x,\varepsilon))}.$$

Thus, for some countable sets $A_\varepsilon, S_\varepsilon$ displaying approximations of A and S, we obtain

$$\int_\Omega f\Big(\frac{\mathrm{d}\mu_\varepsilon}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}\mathscr{L}^n \approx \Big(\sum_{x\in A_\varepsilon} + \sum_{x \in S_\varepsilon}\Big) f\Big(\frac{\mu(\mathrm{B}(x,\varepsilon))}{\mathscr{L}^n(\mathrm{B}(x,\varepsilon))}\Big)\mathscr{L}^n(\mathrm{B}(x,\varepsilon)).$$

We expect the first sum to converge to $\int_\Omega f(\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n})\,\mathrm{d}\mathscr{L}^n$ as $\varepsilon \searrow 0$. Regarding the second term, we note that at $|\mu|^s$-almost all singular points $x \in \Omega$ we have

$$\limsup_{\varepsilon\searrow 0} \frac{|\mu|(\mathrm{B}(x,\varepsilon))}{\mathscr{L}^n(\mathrm{B}(x,\varepsilon))} = +\infty.$$

We then proceed by writing

$$\begin{aligned}\sum_{x\in S_\varepsilon} f\Big(\frac{\mu(\mathrm{B}(x,\varepsilon))}{\mathscr{L}^n(\mathrm{B}(x,\varepsilon))}\Big)\mathscr{L}^n(\mathrm{B}(x,\varepsilon)) = \sum_{x\in S_\varepsilon}\Big(&\frac{\mathscr{L}^n(\mathrm{B}(x,\varepsilon))}{|\mu|(\mathrm{B}(x,\varepsilon))} \times \\ &\times f\Big(\frac{\mu(\mathrm{B}(x,\varepsilon))}{|\mu|(\mathrm{B}(x,\varepsilon)}\frac{|\mu|(\mathrm{B}(x,\varepsilon))}{\mathscr{L}^n(\mathrm{B}(x,\varepsilon))}\Big)|\mu|(\mathrm{B}(x,\varepsilon))\Big).\end{aligned}$$

To understand where the right-hand side might converge to, put $t_\varepsilon := \frac{|\mu|(\mathrm{B}(x,\varepsilon))}{\mathscr{L}^n(\mathrm{B}(x,\varepsilon))}$. We then obtain

$$\begin{aligned}\frac{\mathscr{L}^n(\mathrm{B}(x,\varepsilon))}{|\mu|(\mathrm{B}(x,\varepsilon))} f\Big(\frac{\mu(\mathrm{B}(x,\varepsilon))}{|\mu|(\mathrm{B}(x,\varepsilon)}\frac{|\mu|(\mathrm{B}(x,\varepsilon))}{\mathscr{L}^n(\mathrm{B}(x,\varepsilon))}\Big) &\longrightarrow \lim_{\varepsilon\searrow 0}\frac{1}{t_\varepsilon} f\Big(t_\varepsilon\frac{\mu(\mathrm{B}(x,\varepsilon))}{|\mu|(\mathrm{B}(x,\varepsilon)}\Big) \\ &= \lim_{\substack{s\to\infty \\ \varepsilon\searrow 0}} \frac{1}{s} f\Big(s\frac{\mu(\mathrm{B}(x,\varepsilon))}{|\mu|(\mathrm{B}(x,\varepsilon)}\Big) =: f^\infty\Big(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\Big),\end{aligned}$$

f^∞ being defined in the obvious manner. Hence we expect that

$$\int_\Omega f\Big(\frac{\mathrm{d}\mu_\varepsilon}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}\mathscr{L}^n \approx \int_\Omega f\Big(\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}\mathscr{L}^n + \int_\Omega f^\infty\Big(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|^s}\Big)\,\mathrm{d}|\mu|^s =: f[\mu](\Omega). \tag{2.58}$$

This being only vague heuristics, we yet see that the *recession function* f^∞ takes a pivotal in capturing the integrand's behaviour at infinity, that is, where the measure μ becomes singular for $\mathscr{L}^n$.

We now put the above informal derivation on solid ground. If $f\colon \Omega\times\mathbb{R}^m \to \mathbb{R}\cup\{+\infty\}$ is positively homogeneous of degree one (in the second variable), hence satisfies $f(x,tz) = |t|f(x,z)$ for all $t\in\mathbb{R}$ and $x\in\Omega$, $z\in\mathbb{R}^m$, then we put

$$f[\mu](B) := \int_A f\Big(x,\frac{\mathrm{d}\mu}{\mathrm{d}|\nu|}\Big)\,\mathrm{d}|\nu|,\qquad B\in\mathscr{B}(\Omega) \tag{2.59}$$

whenever $\nu\in\mathscr{M}(\Omega)$ is such that $\mu \ll |\nu|$. By the positive one-homogeneity of f in the second variable, this definition in fact is independent of the specific choice of ν; e.g., $\nu = |\mu|$ will do. Now, if $f\colon\Omega\times\mathbb{R}^m\to\mathbb{R}\cup\{+\infty\}$ is not positively one-homogeneous in the second variable, we extend this definition by introducing the *linear perspective function* $f^\#\colon \Omega\times\mathbb{R}_{\geqslant 0}\times\mathbb{R}^m\to\mathbb{R}$ by

$$f^\#(x,t,\xi) := \begin{cases} tf(x,\frac{z}{t}) & \text{for } t>0\\ f^\infty(x,z) & \text{for } t=0.\end{cases} \tag{2.60}$$

Alluding to above, the *recession function* $f^\infty\colon\Omega\times\mathbb{R}^m\to\mathbb{R}$ is defined by

$$f^\infty(x,z) := \lim_{t\searrow 0} tf\left(x,\frac{z}{t}\right) = \lim_{s\searrow 0}\frac{1}{s}f(x,sz). \tag{2.61}$$

If $f\colon\Omega\times\mathbb{R}^m\to\mathbb{R}$ is continuous, of linear growth and convex in the second variable, then $f^\infty(x,z)$ in fact exists for all $(x,z)\in\Omega\times\mathbb{R}^m$, cf. Lemma 2.51 below. Taking this for granted, we use positive one-homogeneity of $f^\#$ to define

$$f[\mu](B) := \int_B f^\#\Big(x,\frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}\nu},\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu,\qquad B\in\mathscr{B}(\Omega) \tag{2.62}$$

whenever $\nu\in\mathscr{M}(\Omega;\mathbb{R}^m)$ is such that $|\mu|^s+\mathscr{L}^n \ll \nu$. By positive 1-homogeneity of $f^\#$ in the final two variables, this definition in fact is independent of the specific choice of ν, and we shall often take $\nu := \nu_\mu := |\mu|^s + \mathscr{L}^n$ as a reference measure.

To explain how (2.62) is related to (2.58), we denote the Lebesgue-Radon-Nikodým decomposition of μ with respect to $\mathscr{L}^n$ by

$$\mu = \mu^a + \mu^s = \frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}\mathscr{L}^n + \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}|\mu^s|.$$

We then have $\mu^a \ll \mathscr{L}^n$ and $\mu^s\perp\mathscr{L}^n$. Hence there exist disjoint subsets $A,S\subset\Omega$ with $\mathscr{L}^n(S) = |\mu^s|(A) = 0$. Setting $\nu := |\mu^s| + \mathscr{L}^n$, we have $\nu\llcorner A = \mathscr{L}^n\llcorner A$ and $\nu\llcorner S = |\mu^s|\llcorner S$. We split

$$\begin{aligned} f[\mu](B) &= \int_B f^\#\Big(x,\frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}\nu},\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu \\ &= \int_{A\cap B} f^\#\Big(x,\frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}\nu},\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu + \int_{S\cap B} f^\#\Big(x,\frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}\nu},\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu = (*). \end{aligned}$$

Now we note that on A there holds $|\mu^s|\equiv 0$, hence $\nu\llcorner A = \mathscr{L}^n$. Consequently, we have $\frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n} = 1$ and hence $\frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}\nu} = 1$, too. Similarly, on S we have $\mathscr{L}^n\llcorner S = 0$ and hence $\frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}\nu} = \frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}|\mu^s|} = 0$. In this sense, the parameter t in the definition of the linear perspective function is introduced to distinguish conveniently between the absolutely continuous and singular parts of the reference measures. We then obtain, using the positive homogeneity of $f^{\#}$

$$
\begin{aligned}
(*) &= \int_{A\cap B} f^{\#}\Big(x,1,\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu + \int_{S\cap B} f^{\#}\Big(x,0,\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\Big)\,\mathrm{d}\nu \\
&= \int_{A\cap B} f^{\#}\Big(x,1,\frac{\mathrm{d}\mu^a}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}\mathscr{L}^n + \int_{S\cap B} f^{\#}\Big(x,0,\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}\Big)\,\mathrm{d}|\mu^s| \\
&= \int_{A\cap B} f\Big(x,\frac{\mathrm{d}\mu^a}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}\mathscr{L}^n + \int_{S\cap B} f^{\infty}\Big(x,\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}\Big)\,\mathrm{d}|\mu^s| \\
&= \int_{B} f\Big(x,\frac{\mathrm{d}\mu^a}{\mathrm{d}\mathscr{L}^n}\Big)\,\mathrm{d}\mathscr{L}^n + \int_{B} f^{\infty}\Big(x,\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}\Big)\,\mathrm{d}|\mu^s|.
\end{aligned}
$$

Example 2.50 (Integrands m_p)**.** *For $1 \leqslant p < \infty$, we define a convex integrand $\mathrm{m}_p(z) := (1+|z|^p)^{\frac{1}{p}}$. Then we have $\mathrm{m}_p^{\infty}(z) = |z|$, and thus*

$$
\mathrm{m}_p[\mu](B) = \int_B \mathrm{m}_p\left(\frac{\mathrm{d}\mu^a}{\mathrm{d}\mathscr{L}^n}\right)\mathrm{d}\mathscr{L}^n + |\mu^s|(B), \qquad B \in \mathscr{B}(\Omega).
$$

We now give justificiation of some properties used above:

Lemma 2.51. *Let $f\colon \mathbb{R}^m \to \mathbb{R}$ be a convex (hence continuous) function of linear growth: There exist $c_1, c_2 > 0$ and $d \in \mathbb{R}$ such that*

$$
c_1|z| - d \leqslant f(x,z) \leqslant c_2(1+|z|) \qquad \text{for all } (x,z) \in \Omega \times \mathbb{R}^m. \tag{2.63}
$$

Then the following hold:

(a) *f^{∞} is well–defined, convex, homogeneous of degree one and satisfies $c_1|z| \leqslant f^{\infty}(z) \leqslant c_2|z|$ for all $z \in \mathbb{R}^m$.*
(b) *For each $x \in \Omega$, $f^{\#}(x,-,-)\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R}^m \to \mathbb{R}$ is a convex function, homogeneous of degree one and satisfies $c_1|z| \leqslant f^{\#}(t,z) \leqslant c_2(t+|z|)$ for all $(t,z) \in [0,\infty) \times \mathbb{R}^m$.*
(c) *If f is lower semicontinuous, then so are f^{∞} and $f^{\#}$.*

Proof. Ad (a) and (b). Note that, for each $x \in \Omega$, the map $t \mapsto f(x,tz)$ is convex on $\mathbb{R}$ so that for each $z \in \mathbb{R}^m$, the difference quotients $(f(x,tz) - f(x,0))/t$ are monotonically increasing in $t \in \mathbb{R}_{\geqslant 0}$. By (2.63), we deduce for all $z \in \mathbb{R}^{N\times n}$

$$
\frac{f(tz)-f(0)}{t} \leqslant \frac{c_2(1+|tz|)-f(0)}{t} \overset{t\to\infty}{\longrightarrow} c_2|z|
$$

and record that the monotone increase in conjunction with boundedness given by the first of the previous two inqualities gives the existence of $f^{\infty}(z)(= \lim_{t\nearrow\infty}(f(tz) - f(0))/t)$. Now let $(t_1,z_1),(t_2,z_2) \in (0,\infty)\times\mathbb{R}^{N\times n}$ and $0 \leqslant \lambda \leqslant 1$. Then, by convexity of f,

$$
\begin{aligned}
f^{\#}(\lambda t_1 + (1-\lambda)t_2, \lambda z_1 + (1-\lambda)z_2) &\overset{\text{def}}{=} (\lambda t_1 + (1-\lambda)t_2) f\left(\frac{\lambda z_1 + (1-\lambda)z_2}{\lambda t_1 + (1-\lambda)t_2}\right) \\
&\overset{f\text{ convex}}{\leqslant} (\lambda t_1 + (1-\lambda)t_2) f\left(\frac{\lambda t_1}{\lambda t_1 + (1-\lambda)t_2}\frac{z_1}{t_1}\right. \\
&\qquad\left. + \frac{(1-\lambda)t_2}{\lambda t_1 + (1-\lambda)t_2}\frac{z_2}{t_2}\right) \\
&= \lambda f^{\#}(t_1,z_1) + (1-\lambda) f^{\#}(t_2,z_2).
\end{aligned}
$$

Moreover, if $t_1 = 0$ or $t_2 = 0$, we may argue similarly to deduce convexity and hence continuity of f and $f^{\#}$ provided f is assumed continuous, and the claimed homogeneity holds trivially. This settles all assertions apart from the lower semicontinuity part (c), which we tackle now: Let $(t_k, z_k) \to (t, z)$ in $\mathbb{R}_{\geqslant 0} \times \mathbb{R}^{N\times n}$. If $t > 0$, we immediately obtain by lower semicontinuity of f that $f^{\#}(t,z) \leqslant \liminf_{k\to\infty} f^{\#}(t_k, z_k)$. If $t = 0$, then for given $\varepsilon > 0$ we find $s > 0$ satisfying $f^{\infty}(z) \leqslant (f(sz) - c_2)/s + \varepsilon$ and, invoking monotonicity of the difference quotients of f, we obtain (as $s \leqslant 1/t_k$ for all $k \in \mathbb{N}$)

$$
\begin{aligned}
f^{\#}(t,z) &\leqslant \liminf_{k\to\infty} \frac{f(sz_k) - f(0)}{s} + \varepsilon \leqslant \liminf_{k\to\infty} t_k\big(f(z_k/t_k) - f(0)\big) + \varepsilon \\
&\leqslant \liminf_{k\to\infty} f^{\#}(t_k, z_k) + \varepsilon.
\end{aligned}
$$

The proof is complete. $\square$

In view of the direct method and the application of convex functions to measures $\mu = Du$ for $u \in \mathrm{BV}$ later on, we must augment the definitions of the previous section with suitable lower semicontinuity theorems. The first systematic approach to these matters was given by[5] Reshetnyak and Goffman & Serrin, which we recall now in some detail. We begin with

Proposition 2.52 (Reshetnyak)**.** *Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open and let (μ_k) be a sequence of $\mathbb{R}^m$–valued Radon measures of finite total variation which converges to a $\mathbb{R}^m$–valued Radon measure of finite total variation μ on Ω in the weak*–sense. Moreover, assume that all measures μ_k and μ take values in some closed convex cone $K \subset \mathbb{R}^m$. Then the following holds:*

(a) Lower semicontinuity part. *If $\widetilde{f}\colon K \to [0,\infty]$ is a convex, lower semicontinuous and 1-homogeneous function and $\mu_j \overset{*}{\rightharpoonup} \mu$ as $j \to \infty$, then there holds*

$$
\int_{\Omega} \widetilde{f}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\right) \mathrm{d}|\mu| \leqslant \liminf_{j\to\infty} \int_{\Omega} \widetilde{f}\left(\frac{\mathrm{d}\mu_j}{\mathrm{d}|\mu_j|}\right) \mathrm{d}|\mu_j|.
$$

(b) Continuity part. *If $\widetilde{f}\colon K \to [0,\infty)$ is a continuous and 1–homogeneous function and if $\mu_j \to \mu$ strictly*[6] *as $j \to \infty$, then there holds*

$$
\int_{\Omega} \widetilde{f}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\right) \mathrm{d}|\mu| = \lim_{j\to\infty} \int_{\Omega} \widetilde{f}\left(\frac{\mathrm{d}\mu_j}{\mathrm{d}|\mu_j|}\right) \mathrm{d}|\mu_j|.
$$

If $f\colon \mathbb{R}^m \to \mathbb{R}$ is not positively 1-homogeneous, then the preceding proposition is applied to the linear perspective function $f^{\#}$ with respect to the vectorial measures $(\mathscr{L}^n, \mu_j)$ and $(\mathscr{L}^n, \mu)$. We then have

(i) $(\mathscr{L}^n, \mu_j) \overset{*}{\rightharpoonup} (\mathscr{L}^n, \mu)$ if and only if $\mu_j \overset{*}{\rightharpoonup} \mu$, and
(ii) $(\mathscr{L}^n, \mu_j) \to (\mathscr{L}^n, \mu)$ strictly if and only if $\mu_j \overset{*}{\rightharpoonup} \mu$ and

$$
\int_{\Omega} \sqrt{1 + \left|\frac{\mathrm{d}\mu_j^a}{\mathrm{d}\mathscr{L}^n}\right|^2}\, \mathrm{d}\mathscr{L}^n + |\mu_j^s|(\Omega) \to \int_{\Omega} \sqrt{1 + \left|\frac{\mathrm{d}\mu^a}{\mathrm{d}\mathscr{L}^n}\right|^2}\, \mathrm{d}\mathscr{L}^n + |\mu^s|(\Omega)
$$

as $j \to \infty$.

[5]References are to be added soon.

[6]In the sense that $\mu_j \overset{*}{\rightharpoonup} \mu$ and $|\mu_j|(\Omega) \to |\mu|(\Omega)$ as $j \to \infty$.

Hence, if we specify to measures $\mu = Du$, then (ii) is implied by the area-strict convergence. We thus deduce from Proposition 2.52

Proposition 2.53. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Moreover, let $f\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function of linear growth which is convex in its second variable. Then the integral function*

$$\mathscr{F}[u] := \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{\mathrm{d}\mathrm{D}^s u}{\mathrm{d}|\mathrm{D}^s u|}\Big)\,\mathrm{d}|\mathrm{D}^s u|, \qquad u \in \mathrm{BV}(\Omega;\mathbb{R}^N),$$

is lower semicontinuous with respect to weak-convergence on* $\mathrm{BV}(\Omega;\mathbb{R}^N)$*, and continuous with respect to area-strict convergence on* $\mathrm{BV}(\Omega;\mathbb{R}^N)$.

Let us note that the preceding theory does not easily extend to more general convexity regimes, and we shall come back to this issue below.

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We now specify the general framework outlined in the previous subsections to the Dirichlet problem on BV. As usual, we put

$$\mathscr{F}[v;\Omega] := \int_\Omega f(x, Dv)\,\mathrm{d}x, \tag{2.64}$$

where $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary. We then find $R > 0$ so large such that $\mathrm{B}(0,R) \Subset \Omega$. Given a Dirichlet datum $u_0 \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$, we find a map $v_0 \in \mathrm{W}^{1,1}(\mathrm{B}(0,R)\setminus\overline{\Omega};\mathbb{R}^N)$ such that $\mathrm{Tr}_{\partial\mathrm{B}(0,R)}(v_0) = 0$ and $\mathrm{Tr}_{\partial\Omega}(v_0) = \mathrm{Tr}_{\partial\Omega}(u_0)$ $\mathscr{H}^{n-1}$-a.e. on $\partial\mathrm{B}(0,R)$ and $\partial\Omega$, respectively. Given $v \in \mathrm{BV}(\Omega;\mathbb{R}^N)$, we put

$$\widetilde{v}(x) := \begin{cases} v(x) & \text{for } x \in \Omega, \\ v_0(x) & \text{for } x \in \Omega \setminus \overline{\mathrm{B}(0,R)}. \end{cases} \tag{2.65}$$

By the gluing theorem, Theorem 2.49, $\widetilde{v} \in \mathrm{BV}(\mathrm{B}(0,R);\mathbb{R}^N)$ together with

$$\begin{aligned} D\widetilde{v} = \nabla v\mathscr{L}^n \llcorner \Omega + D^s v \llcorner \Omega + (\mathrm{Tr}_{\partial\Omega}(u_0 - v) \odot \nu_{\partial\Omega})\mathscr{H}^{n-1} \llcorner \partial\Omega \\ + \nabla v_0 \mathscr{L}^n \llcorner (\mathrm{B}(0,R) \setminus \overline{\Omega}). \end{aligned}$$

Given a continuous integrand $f\colon \mathbb{R}^n \times \mathbb{R}^{N\times n} \to \mathbb{R}$ which is convex and of linear growth in its second variable, we may thus record

$$\begin{aligned} f[D\widetilde{v}](\mathrm{B}(0,R)) &= \int_\Omega f(\nabla v)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{\mathrm{d}D^s v}{\mathrm{d}|D^s v|}\Big)\,\mathrm{d}|D^s v| \\ &+ \int_{\partial\Omega} f^\infty(x, \mathrm{Tr}_{\partial\Omega}(u_0 - v) \otimes \nu_{\partial\Omega})\,\mathrm{d}\mathscr{H}^{n-1} + \int_{\mathrm{B}(0,R)\setminus\overline{\Omega}} f(\nabla v_0)\,\mathrm{d}x \end{aligned} \tag{2.66}$$

and consider the variational principle

$$\text{to minimise } \ \overline{\mathscr{F}}[v;\mathrm{B}(0,R)] := f[D\widetilde{v}](\mathrm{B}(0,R)) \ \text{ over all } v \in \mathrm{BV}(\Omega;\mathbb{R}^N). \tag{2.67}$$

The main theorem of this section then reads as follows.

Theorem 2.54 (Existence of minima for the relaxed problem)**.** *Let $f\colon \mathbb{R}^n \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous integrand which is convex in its second variable and satisfies*

$$c_1|z| - c_2 \leqslant f(x,z) \leqslant c_3(1 + |z|) \qquad \text{for all } (x,z) \in \mathbb{R}^n \times \mathbb{R}^{N\times n}. \tag{2.68}$$

Then the following hold:

(a) *The variational problem* (2.67) *possesses a solution* $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$.

(b) *Put, for $v \in \mathrm{BV}(\Omega;\mathbb{R}^N)$,*

$$
\begin{aligned}
\overline{\mathscr{F}}_{u_0}[v;\Omega] := \int_\Omega f(\nabla v)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{\mathrm{d}D^s v}{\mathrm{d}|D^s v|}\Big)\,\mathrm{d}|D^s u| \\
+ \int_{\partial\Omega} f^\infty(\mathrm{Tr}_{\partial\Omega}(u_0 - v)\otimes \nu_{\partial\Omega})\,\mathrm{d}\mathscr{H}^{n-1}.
\end{aligned} \tag{2.69}
$$

Then $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ is a minimiser of $\overline{\mathscr{F}}_{u_0}[-;\Omega]$ over $\mathrm{BV}(\Omega;\mathbb{R}^N)$ if and only if u is a minimiser of $\overline{\mathscr{F}}[-;\mathrm{B}(0,R)]$ over $\mathrm{BV}(\Omega;\mathbb{R}^N)$. In particular, by (a)*, there exists a minimiser $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ of $\overline{\mathscr{F}}_{u_0}[-;\Omega]$.*

(c) Absence of gaps: *Defining $\mathscr{F}$ by* (2.64)*, we have*

$$
\inf \mathscr{F}[\mathrm{W}^{1,1}_{u_0}(\Omega;\mathbb{R}^N)] = \min \overline{\mathscr{F}}_{u_0}[\mathrm{BV}(\Omega;\mathbb{R}^N);\Omega] = \inf \overline{\mathscr{F}}_{u_0}[\mathrm{BV}(\Omega;\mathbb{R}^N);\Omega]. \tag{2.70}
$$

Proof. We divide the proof into three steps.

Ad (a). Since $0 < R < \infty$, we deduce from (2.68) that $\overline{\mathscr{F}}[-;\mathrm{B}(0,R)]$ is bounded from below on $\mathrm{BV}(\Omega;\mathbb{R}^N)$. Thus, $m := \inf \overline{\mathscr{F}}[\mathrm{BD}(\Omega)] > -\infty$ and we find $(u_j) \subset \mathrm{BV}(\Omega;\mathbb{R}^N)$ such that $\overline{\mathscr{F}}[u_j;\mathrm{B}(0,R)] \to m$. By the compactness principle on BV – Theorem 2.37 – we find $w \in \mathrm{BV}(\mathrm{B}(0,R);\mathbb{R}^N)$ and a subsequence $(u_{j(i)}) \subset (u_{j(i)})$ such that

$$
\widetilde{u}_{j(i)} \overset{*}{\rightharpoonup} w \qquad \text{in } \mathrm{BV}(\mathrm{B}(0,R);\mathbb{R}^N).
$$

Let us record in advance that w coincides with v_0 in $\mathrm{B}(0,R)\setminus\overline{\Omega}$. Then, by the lower semicontinuity part of Proposition 2.53 we deduce that

$$
\begin{aligned}
f[Dw](\Omega) &+ \int_{\partial\Omega} f^\infty(x,\mathrm{Tr}_{\partial\Omega}(u_0 - w)\otimes\nu_{\partial\Omega})\,\mathrm{d}\mathscr{H}^{n-1} + \int_{\mathrm{B}(0,R)\setminus\overline{\Omega}} f(\nabla v_0)\,\mathrm{d}x \\
&= f[Dw](\mathrm{B}(0,R)) \\
&\leqslant \liminf_{i\to\infty} f[D\widetilde{u}_{j(i)}](\mathrm{B}(0,R)) \\
&= \liminf_{i\to\infty} \Big(f[Du_{j(i)}](\Omega) + \int_{\partial\Omega} f^\infty(x,\mathrm{Tr}_{\partial\Omega}(u_0 - u_{j(i)})\otimes\nu_{\partial\Omega})\,\mathrm{d}\mathscr{H}^{n-1}\Big) \\
&\qquad\qquad + \int_{\mathrm{B}(0,R)\setminus\overline{\Omega}} f(\nabla v_0)\,\mathrm{d}x = m.
\end{aligned}
$$

Consequently, w is a solution of the variational principle (2.67), which is (a).

Ad (b). By (2.66), we have for all $v \in \mathrm{BV}(\Omega;\mathbb{R}^N)$

$$
\overline{\mathscr{F}}_{u_0}[v;\Omega] + \int_{\mathrm{B}(0,R)\setminus\overline{\Omega}} f(\nabla v_0)\,\mathrm{d}x = \overline{\mathscr{F}}[\widetilde{v};\mathrm{B}(0,R)].
$$

Since the second term on the left-hand side is constant, this immediately yields (b).

Ad (c). The second equation restates the fact that the minimisation of $\overline{\mathscr{F}}_{u_0}[-;\Omega]$ over $\mathrm{BV}(\Omega;\mathbb{R}^N)$ possesses a solution. To establish the first equation, note that

$$
\overline{\mathscr{F}}_{u_0}|_{\mathrm{W}^{1,1}_{u_0}(\Omega;\mathbb{R}^N)} = \mathscr{F}|_{\mathrm{W}^{1,1}_{u_0}(\Omega;\mathbb{R}^N)}.
$$

Hence we obtain $\inf \mathscr{F}[\mathrm{W}^{1,1}_{u_0}(\Omega;\mathbb{R}^N)] \geqslant \min \overline{\mathscr{F}}_{u_0}[\mathrm{BV}(\Omega;\mathbb{R}^N);\Omega]$. For the reverse inequality, let $u \in \mathrm{BD}(\Omega)$ be a minimiser for $\overline{\mathscr{F}}_{u_0}[-;\Omega]$ so that, by (b), $\widetilde{u}$ is a solution of the auxiliary variational principle (2.67). Smoothly approximating u with maps in $u_0 + \mathrm{C}^\infty_c(\Omega;\mathbb{R}^N)$ in the area-strict topology, we find $(u_j) \subset u_0 + \mathrm{C}^\infty_c(\Omega;\mathbb{R}^N)$ such that $\widetilde{u}_j \to \widetilde{u}$ symmetric-area strictly in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ as $j\to\infty$. Then, applying the Reshetnyak continuity theorem,

$$\begin{aligned}\min\overline{\mathscr{F}}[\mathrm{BV}(\Omega;\mathbb{R}^N);\mathrm{B}(0,R)] &= \overline{\mathscr{F}}[\widetilde{u};\mathrm{B}(0,R)] = \lim_{j\to\infty}\overline{\mathscr{F}}[\widetilde{u}_j;\mathrm{B}(0,R)] \\ &\geqslant \inf\mathscr{F}[\mathrm{W}^{1,1}_{u_0}(\Omega;\mathbb{R}^N)] + \int_{\mathrm{B}(0,R)\setminus\overline{\Omega}} f(x,\mathscr{E}v_0)\,\mathrm{d}x.\end{aligned} \tag{2.71}$$

In the ultimate step, we have used that $\mathrm{Tr}_{\partial\Omega}(\widetilde{u}_j)=\mathrm{Tr}_{\partial\Omega}(u_0)$ $\mathscr{H}^{n-1}$-a.e. on $\partial\Omega$. Now, again by (b),

$$\min\overline{\mathscr{F}}[\mathrm{BV}(\Omega;\mathbb{R}^N);\mathrm{B}(0,R)] = \min\overline{\mathscr{F}}_{u_0}[\mathrm{BV}(\Omega;\mathbb{R}^N);\Omega] + \int_{\mathrm{B}(0,R)\setminus\overline{\Omega}} f(x,\nabla v_0)\,\mathrm{d}x.$$

Combining the last equation with (2.71), we arrive at

$$\inf\mathscr{F}[\mathrm{W}^{1,1}_{u_0}(\Omega;\mathbb{R}^N)] \leqslant \min\overline{\mathscr{F}}_{u_0}[\mathrm{BV}(\Omega;\mathbb{R}^N);\Omega],$$

and the proof is complete. □

Assertion (c) is referred to as a *no gap result.* In fact, when passing the extension of $\mathscr{F}$ to $\mathrm{BV}(\Omega;\mathbb{R}^N)$ subject to the given Dirichlet data, it is in principle possible that the relevant extension or relaxation attains a minimum which is strictly smaller than the infimum of the original problem – and thus reveals a gap. In this sense, (c) asserts that such a gap does not occur in our framework.

Next we wish to study the regularity of the Dirichlet problem. Our primary goal is then to find

criteria which ensure the $\mathrm{W}^{1,1}$*-regularity of all generalised minima.*

To this end, it is – in some sense – compulsory to consider integrands which are elliptic in the sense of strict convexity. This is manifested by the following example.

Example 2.55. *We consider now the standard variational integral with* $f=|\cdot|$, $\Omega=(-1,1)$ *and prescribe the Dirichlet values* $u(-1)=u_0(-1)=-1$, $u(1)=u_0(1)=1$. *Then we have* $\inf_{\mathrm{W}^{1,1}_{u_0}(\Omega)} F = 2$. *In fact, let* $v\in\mathrm{W}^{1,1}_{u_0}(\Omega)$ *so that* v *is absolutely continuous. Hence we have*

$$2 = |v(1)-v(-1)| = \left|\int_\Omega v'(t)\,\mathrm{d}t\right| \leqslant \int_\Omega |v'(t)|\,\mathrm{d}t = \int_\Omega |\nabla v(t)|\,\mathrm{d}t.$$

On the other hand, consider the sequence $(v_k)\subset\mathrm{W}^{1,1}_{u_0}(\Omega)$ *given by*

$$v_k(t) = \mathbb{1}_{[-1,-1/k]}(t) + \mathbb{1}_{(-1/k,1/k)}(t)kt + \mathbb{1}_{[1/k,1]}(t).$$

Then $v_k'(t)=\mathbb{1}_{(-1/k,1/k)}k$ *and hence* $F[v_k;\Omega]=2$ *for all* $k\in\mathbb{N}$. *In consequence,* each v_k *is* F*-minimising and so is its weak*-limit in* $\mathrm{BV}(\Omega)$.

Even though it is possible to develop a theory of how to derive Euler-Lagrange equations for BV-minimisers, it is difficult to implement this setting when aiming for higher differentiability estimates. The Euler-Lagrange equation for measures contains the recession function F^∞ of the integrands, and F^∞ ignores the ellipticity properties of F: As an example, recall that the recession function of $\mathrm{m}_p^\infty(z)=|z|$ regardless of $1<p<\infty$, where $\mathrm{m}_p(z)=(1+|z|^p)^{1/p}$. Hence, we aim to work on suitable minimising sequences obtained via stabilised functionals and establish uniform regularity estimates; doing so, we may work on $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ and exploit the ellipticity properties of the integrands F.

Toward this objective, a standard device is to perform viscosity approximations of a given minimiser by adding the Dirichlet energy to the functional: Let F be an h-elliptic, linear growth integrand. Then, for $j \in \mathbb{N}$, we consider $\mathscr{F}_j$ given by

$$\mathscr{F}_j[u] := \int_\Omega F(\nabla u)\,\mathrm{d}x + \frac{1}{2j}\int_\Omega |\nabla u|^2\,\mathrm{d}x.$$

Assuming that $u_0 \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$, these functionals are defined on $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$, and because of the additional Dirichlet energy term, enjoy better ellipticity properties than $\mathscr{F}$ itself. In particular, if we pass to the Euler-Lagrange equation satisfied by each u_k, the Dirichlet integral converts into a weak Laplacian and thus makes the classical elliptic estimate available (even though not uniformly in j). Note that $\mathscr{F}_j$ is a strictly convex functional and hence its minima are unique; so, for each $j \in \mathbb{N}$ there exists a unique minimiser v_j of $\mathscr{F}_j$ as $\mathscr{F}_j$ is easily shown to be sequentially lower semicontinuous with respect to weak convergence on $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$. In this way, we obtain a *vanishing viscosity* sequence $(v_j) \subset \mathrm{W}^{1,2}_{u_0}(\Omega;\mathbb{R}^N)$. Our aim is to prove that (v_j) converges in a suitable sense to a map $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ which, in turn, is a generalised minimiser for $\mathscr{F}$ subject to the Dirichlet datum u_0.

Here we encounter three major obstructions:

(i) *Approximation of the boundary values.* As $\mathrm{L}^1(\partial\Omega;\mathbb{R}^N)$ is the trace space of both $\mathrm{BV}(\Omega;\mathbb{R}^N)$ and $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$, the correct Dirichlet classes to work with are given by the affine space $\mathscr{D}_{u_0} := \mathrm{W}^{1,1}_{u_0}(\Omega;\mathbb{R}^N) := u_0 + \mathrm{W}^{1,1}_0(\Omega;\mathbb{R}^N)$ instead of $\mathrm{W}^{1,2}_{u_0}(\Omega;\mathbb{R}^N)$. To reduce to the latter case, we may employ an extension and regularisation procedure for the boundary values. This matter primarily is of technical nature and we refer to the end of this subsection for its resolution.

(ii) *The regularity estimates only apply to one particular generalised minimiser.* When establishing uniform regularity estimates on the single members of the viscosity approximation sequence (v_j) (for instance, uniform local $L\log L$-bounds on their gradients), these estimate will only carry over to the weak*-limit of *this specific viscosity approximation sequence*. This is due to the fact that even though $\mathscr{F}$ is strictly convex on $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$, the relaxed functional $\overline{\mathscr{F}}_{u_0}$ is *not* strictly convex on BV. To be more precise, recall that if $\mathscr{F}$ is strictly convex on $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ and $\mathscr{G}$ is a convex functional on $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$, then $\mathscr{F}+\mathscr{G}$ is strictly convex on $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$. This is so because both $\mathscr{F}$ and $\mathscr{G}$ act on the same part of the gradient, namely, the full one. When we pass to the relaxed functional

$$\overline{\mathscr{F}}_{u_0}[u] = \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty\Big(\frac{\mathrm{d}\mathrm{D}^s u}{\mathrm{d}|\mathrm{D}^s u|}\Big)\,\mathrm{d}|\mathrm{D}^s u| + \int_{\partial\Omega} f^\infty(\mathrm{Tr}_{\partial\Omega}(u_0-u)\otimes\nu_{\partial\Omega})\,\mathrm{d}\mathscr{H}^{n-1}$$

for $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$, then the first integral term is convex, the ultimate two being convex. However, as these terms act on mutually singular parts of the gradients, we *cannot* assert the overall strict convexity of $\overline{\mathscr{F}}_{u_0}$ and hereafter the uniqueness of generalised minima. As quickly alluded to above, the two integrals containing f^∞ give rise to two different sources of non-uniqueness:

- Non-vanishing of the singular part $\mathrm{D}^s u$ in the interior of $\overline{\Omega}$,
- Non-attainment of the correct boundary values u_0 on $\partial\Omega$.

Thus, should we be in position to establish higher regularity for one generalised minimiser, the plain viscosity strategy from above merely applies to one generalised minimiser; others, perhaps more irregular ones might exist.

Our primary aim thus is to establish that the singular parts of *all* generalised minima vanish. In some sense, this is a *local* regularity result, but needs to be approached differently from the plain viscosity strategy from above. For the Dirichlet problem under consideration, the only contributions available so far are able to deal with a-elliptic integrands $f\colon \mathbb{R}^{N\times n}\to\mathbb{R}$, $1<a\leqslant 3$. When such a result is achieved, generalised minima do not need to be unique in general either, the reason being the boundary integral containing the recession function.

(iii) *Boundary behaviour of generalised minima.* Should we be able to achieve $\mathrm{W}^{1,1}$-regularity of all generalised minima, there are only very seldom instances of full uniqueness of generalised minima. This is in line with the counterexamples to uniqueness to be outlined below. In this situation, however, we strive for describing the set of generalised minima. As we shall see, if generalised minima are unique up to constants (or, in the more advanced symmetric gradient case, rigid deformations), then $\mathrm{GM}(\mathscr{F};u_0)$ is essentially an interval in $\mathrm{BV}(\Omega;\mathbb{R}^N)$, respectively.

To streamline terminology, we will call regularity estimates that apply to all generalised minima and not only to a particular one *universal*. The aim of the remaining section hence is to outline approach that yields universal $\mathrm{W}^{1,1}$-regularity estimates for all generalised minima. To this end, we crucially utilise the variational principle due to Ekeland:

Theorem 2.56 (Ekeland Variational Principle)**.** *Let (X,d) be a complete metric space and let $\mathcal{F}\colon X\to\overline{\mathbb{R}}$ be a lower semicontinuous functional (with respect to the metric topology induced by d) which moreover is bounded from below and attains a finite value at some $x_0\in X$. Assume that for some $u\in X$ and some $\varepsilon>0$ there holds*

$$\mathcal{F}[u]\leqslant\inf_X\mathcal{F}+\varepsilon.$$

Then there exists $v\in X$ such that

(a) $d(u,v)\leqslant 1$,
(b) $\mathcal{F}[v]\leqslant\mathcal{F}[u]$,
(c) $\mathcal{F}[v]\leqslant\mathcal{F}[u]+\varepsilon d(v,w)$ *for all* $w\in X$.

Proof. We construct the element $v\in X$ as the limit of a particular Cauchy sequence (u_k) which is defined inductively as follows. We put $u_1:=u$. Assuming that for $k\in\mathbb{N}$ the members $u_1,...,u_k$ have been constructed, we note that the set $S_k:=\{w\in X\colon\ \mathcal{F}[w]\leqslant\mathcal{F}[u_k]-\varepsilon d(u_k,w)\}$ contains u_k and thus is non–empty. In conclusion, we find $u_{k+1}\in S_k$ such that

$$\mathcal{F}[u_{k+1}]\leqslant\frac{1}{2}\Big(\mathcal{F}[u_k]+\inf_{S_k}\mathcal{F}\Big). \tag{2.72}$$

We claim that (u_k) is d–Cauchy. Indeed, fix $k\in\mathbb{N}$ and note that, since $u_{k+1}\in S_k$, there holds

$$\varepsilon d(u_{k+1},u_k)\leqslant\mathcal{F}[u_k]-\mathcal{F}[u_{k+1}]. \tag{2.73}$$

Consequently, a telescope sum argument yields for all $m\in\mathbb{N}$

$$\varepsilon d(u_{k+m},u_k)\leqslant\varepsilon\sum_{i=1}^m d(u_{k+i},u_{k+i-1})\leqslant\mathcal{F}[u_k]-\mathcal{F}[u_{k+m}]. \tag{2.74}$$

However, since $(\mathcal{F}[u_k])$ is decreasing because of (2.73) and is bounded below by assumption, it converges to some $a \in \mathbb{R}$. Hence, (2.74) yields that (u_k) is d–Cauchy and therefore, by completeness of (X, d), converges to some $v \in X$. Lower semicontinuity of $\mathcal{F}$ consequently gives $\mathcal{F}[v] \leqslant \liminf_{k\to\infty} \mathcal{F}[u_k] = a$ and hence, sending $m \to \infty$ in (2.74), $\varepsilon d(v, u_k) \leqslant \mathcal{F}[u_k] - \mathcal{F}[v]$. Setting $k = 1$ then yields

$$0 \leqslant \varepsilon d(u, v) \leqslant \mathcal{F}[u] - \mathcal{F}[v] \leqslant \mathcal{F}[u] - \inf_X \mathcal{F} \leqslant \varepsilon$$

so that (a) and (b) follow. In view of (c), we argue by contradiction and hereafter suppose that there exists $w \in X$ such that $\mathcal{F}[w] < \mathcal{F}[v] - \varepsilon d(w, v)$. Then, since $\mathcal{F}[v] \leqslant \mathcal{F}[u_k] - \varepsilon d(u_k, v)$, we obtain by the triangle inequality

$$\mathcal{F}[w] < \mathcal{F}[v] - \varepsilon d(w, v) \leqslant \mathcal{F}[u_k] - \varepsilon(d(u_k, v) + d(v, w)) \leqslant \mathcal{F}[u_k] - \varepsilon d(u_k, w),$$

and so we conclude that for each $k \in \mathbb{N}$ we have $w \in S_k$. Therefore, we have $\inf_{S_k} \mathcal{F} \leqslant \mathcal{F}[w]$ for all $k \in \mathbb{N}$ and thus, by (2.72),

$$2\mathcal{F}[u_{k+1}] - \mathcal{F}[u_k] \leqslant \inf_{S_k} \mathcal{F} \leqslant \mathcal{F}[w] < \mathcal{F}[v] - d(v, w).$$

Passing to the limit $k \to \infty$ in the preceding inequality yields $\mathcal{F}[v] \leqslant \mathcal{F}[w] < \mathcal{F}[v] - \varepsilon d(v, w)$ which is impossible. The proof is complete. □

As a simple consequence of the preceding theorem, we obtain the following corollary.

Corollary 2.57 ([42, Rem. 5.5])**.** *Let (X, d) be a complete metric space and let $\mathcal{F}\colon X \to \overline{\mathbb{R}}$ be a lower semicontinuous functional (with respect to the metric topology induced by d) which moreover is bounded from below and attains a finite value at some $x_0 \in X$. Assume that for some $u \in X$ and some $\varepsilon > 0$ there holds*

$$\mathcal{F}[u] \leqslant \inf_X \mathcal{F} + \varepsilon.$$

Then there exists $v \in X$ such that

(a) $d(u, v) \leqslant \sqrt{\varepsilon}$,
(b) $\mathcal{F}[v] \leqslant \mathcal{F}[u]$,
(c) $\mathcal{F}[v] \leqslant \mathcal{F}[u] + \sqrt{\varepsilon} d(v, w)$ *for all* $w \in X$.

Proof. Define a new metric by $\widetilde{d} := d/\sqrt{\varepsilon}$. Then the claim follows immediately from the Ekeland variational principle. □

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To implement the Ekeland variational principle with respect to sufficiently weak perturbations, we introduce certain negative Sobolev spaces as follows. Given $k \in \mathbb{N}$, we define the space $\mathrm{W}^{-k,1}(\Omega; \mathbb{R}^n)$ as follows:

$$\mathrm{W}^{-k,1}(\Omega; \mathbb{R}^n) := \Big\{ T \in \mathscr{D}'(\Omega; \mathbb{R}^n)\colon\; T = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \leqslant k}} \partial^\alpha T_\alpha,\; T_\alpha \in \mathrm{L}^1(\Omega; \mathbb{R}^n) \text{ for all } |\alpha| \leqslant k \Big\}.$$

The linear space $\mathrm{W}^{-k,1}(\Omega; \mathbb{R}^n)$ is canonically endowed with the norm

$$\|T\|_{\mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)} := \inf \sum_{|\alpha| \leqslant k} \|T_\alpha\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)}, \tag{2.75}$$

the infimum ranging over all representations $T = \sum_{|\alpha| \leqslant k} \partial^\alpha T_\alpha$ with $T_\alpha \in \mathrm{L}^1(\Omega; \mathbb{R}^n)$. This is the canonical higher order version of the space $\mathrm{W}^{-1,1}(\Omega; \mathbb{R}^n)$ as introduced in [19]. We collect its most important properties in the following lemma.

Lemma 2.58. *Let $\Omega \subset \mathbb{R}^n$ be open and let $k \in \mathbb{N}$ be given. Then the following holds:*

(a) $(\mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n), \|\cdot\|_{\mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)})$ *is a Banach space.*

(b) *For every $u \in \mathrm{L}^1(\Omega;\mathbb{R}^n)$ and every $\beta \in \mathbb{N}_0^n$ with $|\beta| \leqslant k$ there holds*

$$\|\partial^\beta u\|_{\mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)} \leqslant \|u\|_{\mathrm{W}^{|\beta|-k,1}(\Omega;\mathbb{R}^n)}.$$

Proof. For (a), we closely follow [19] and consider the mapping $\Phi\colon \mathrm{L}^1(\Omega;\mathbb{R}^n)^N \ni (T_\alpha)_{|\alpha|\leqslant k} \mapsto \sum_{|\alpha|\leqslant k} \partial^\alpha T_\alpha \in \mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)$, where $N := \#\{\alpha \in \mathbb{N}_0^n\colon\ |\alpha| \leqslant k\}$. By definition of $\mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)$, Φ is a bounded linear operator and thus $\ker(\Phi)$ is a Banach space in itself. By definition of the quotient norm, the canonical quotient map $\Psi\colon \mathrm{L}^1(\Omega;\mathbb{R}^n)^N/\ker(\Phi) \to \mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)$ is surjective and isometric. Thus, as $\ker(\Phi)$ is Banach, so is $\mathrm{L}^1(\Omega;\mathbb{R}^n)^N/\ker(\Phi)$ and eventually, as the isometric image of a Banach space, $(\mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n), \|\cdot\|_{\mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)})$. For (b), let $\varepsilon > 0$ and choose $(T_\alpha)_\alpha \in \mathrm{L}^1(\Omega;\mathbb{R}^n)^N$ such that $u = \sum_{|\alpha|\leqslant k-|\beta|} \partial^\alpha T_\alpha$ and

$$\sum_{|\alpha|\leqslant k-|\beta|} \|T_\alpha\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \leqslant \|u\|_{\mathrm{W}^{|\beta|-k,1}(\Omega;\mathbb{R}^n)} + \varepsilon.$$

On the other hand, $\partial^\beta u = \sum_{|\alpha|\leqslant k-|\beta|} \partial^{\alpha+\beta} T_\alpha =: \sum_{|\gamma|\leqslant k} \partial^\gamma S_\gamma$, where $S_\gamma = T_\alpha$ if $\gamma = \alpha+\beta$ for some $|\alpha| \leqslant k-\beta$ and $S_\gamma = 0$ otherwise. Therefore,

$$\|\partial^\beta u\|_{\mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)} \leqslant \sum_{|\gamma|\leqslant k} \|S_\gamma\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \leqslant \sum_{|\alpha|\leqslant k-|\beta|} \|T_\alpha\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \leqslant \|u\|_{\mathrm{W}^{|\beta|-k,1}(\Omega;\mathbb{R}^n)} + \varepsilon,$$

and we then send $\varepsilon \searrow 0$ to conclude the proof. $\square$

Next, a lower semicontinuity result in the spirit of [**?**, Lem. 3.2], [19, Lem. 2.6]:

Lemma 2.59. *Let $1 < q < \infty$, $k \in \mathbb{N}$ be given and let Ω be open and bounded with Lipschitz boundary $\partial\Omega$. Suppose that $\mathfrak{f}\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is a convex function that satisfies $c^{-1}|z|^q - d \leqslant \mathfrak{f}(z) \leqslant c(1+|z|^q)$ for some $c,d > 0$ and all $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. Then, for every $u_0 \in \mathrm{W}^{1,q}(\Omega;\mathbb{R}^n)$, the functional*

$$\mathcal{F}[u] := \begin{cases} \displaystyle\int_\Omega \mathfrak{f}(\nabla u)\,\mathrm{d}x & \text{if } u \in \mathscr{D}_{u_0} := u_0 + \mathrm{W}^{1,q}_0(\Omega;\mathbb{R}^n), \\ +\infty & \text{if } u \in \mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n) \setminus \mathscr{D}_{u_0} \end{cases}$$

is lower semicontinuous for the norm topology on $\mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)$.

Proof. Let $g, g_1, g_2, \ldots \in \mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)$ be such that $g_m \to g$ with respect to the norm topology on $\mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)$. If $\liminf_{m\to\infty} \mathcal{F}[g_m] = +\infty$, there is nothing to prove. Hence assume without loss of generality that $\lim_{j\to\infty} \mathcal{F}[g_{m(j)}] = \liminf_{m\to\infty} \mathcal{F}[g_m] < \infty$. Then necessarily $g_{m(j)} \in \mathscr{D}_{u_0}$ for all sufficiently large indices j and, since $c^{-1}|z|^q - d \leqslant \mathfrak{f}(z)$ for all $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, we obtain that $(g_{m(j)})$ is bounded in $\mathrm{W}^{1,q}(\Omega;\mathbb{R}^n)$. Since $1 < q < \infty$, there exists a subsequence $(g_{m(j(i))}) \subset (g_{m(j)})$ which converges weakly in $\mathrm{W}^{1,q}(\Omega;\mathbb{R}^n)$ to some $\widetilde{g} \in \mathscr{D}_{u_0}$ (note that $\mathscr{D}_{u_0}$ is weakly closed in $\mathrm{W}^{1,q}(\Omega;\mathbb{R}^n)$). By the Rellich-Kondrachov theorem, we can moreover assume that $g_{m(j(i))} \to \widetilde{g}$ strongly in $\mathrm{L}^q(\Omega;\mathbb{R}^n)$. Then, since $\mathrm{L}^q(\Omega;\mathbb{R}^n) \hookrightarrow \mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)$ by Lemma 2.58(b),

$$\|g - \widetilde{g}\|_{\mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)} \leqslant \|g - g_{m(j(i))}\|_{\mathrm{W}^{-k,1}(\Omega;\mathbb{R}^n)} + \|\widetilde{g} - g_{m(j(i))}\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \to 0$$

as $i \to \infty$, and thus $g = \widetilde{g}$. By standard results on lower semicontinuity of convex variational integrals of superlinear growth (or, alternatively, Reshetnyak's lower semicontinuity theorem) $\nabla g_{m(j(i))}\mathscr{L}^n \overset{*}{\rightharpoonup} \nabla g \mathscr{L}^n$ thus yields

$$\mathcal{F}[g] \leqslant \liminf_{i\to\infty} \mathcal{F}[g_{m(j(i))}] = \liminf_{m\to\infty} \mathcal{F}[g_m].$$

The proof is complete. □

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2.6. **The scale of a-elliptic integrands.** For the $\mathrm{W}^{1,1}$-regularity for the Dirichlet problem, it is instructive to consider the important instance of power-like decay behaviour for the second derivatives first. Let $1 < a < \infty$. Inspired by Bernstein's work on minimal surface type problems [14], we call F with slight abuse of notation *a-elliptic* if and only if F is h-elliptic with $h(t) = (1+t^2)^{-\frac{a}{2}}$; that is, there exist $0 < \lambda \leqslant \Lambda < \infty$ such that

$$\lambda \frac{|z|^2}{(1+|\xi|^2)^{a/2}} \leqslant \langle F''(\xi)z, z\rangle \leqslant \Lambda \frac{|z|^2}{(1+|\xi|^2)^{1/2}} \qquad \text{for all } z,\xi \in \mathbb{R}^{N\times n}. \tag{2.76}$$

This condition also goes by the name of *μ-ellipticity* (where $\mu \equiv a$ in our terminology), a notion introduced by Bildhauer & Fuchs based on their previous work with Mingione in the context of (p,q)-type problems. The a-elliptic integrands form a *scale* in the sense that if $1 < a_1 \leqslant a_2 < \infty$ and F is a_1-elliptic, then it is a_2-elliptic, too. Its significance for linear growth problems is based on the fact that it helps to identify various borderline cases in a concise way, and we shall see in the course of the chapter that $a = 3$ is such a borderline ellipticity. Before passing to examples that show the richness of this scale, let us briefly comment on why $a = 1$ is *explicitely excluded* in the above definition of a-ellipticity.

For simplicity, suppose that $N = n = 1$ and $F \in \mathrm{C}^2(\mathbb{R})$ satisfies (2.76) with $a = 1$ and $F'(0) = 0$. Then we integrate to find for $\xi > 0$ in this situation that

$$\begin{aligned} F'(\xi) = (F'(\xi) - F'(0)) + F'(0) &= \int_0^1 \langle F''(t\xi), \xi\rangle \,\mathrm{d}t + F'(0) \\ &= \int_0^1 \frac{\xi}{(1+|t|^2\,\xi^2)^{\frac{1}{2}}}\,\mathrm{d}t = \int_0^\xi \frac{\mathrm{d}t}{(1+t^2)^{\frac{1}{2}}} \overset{t\to\infty}{\longrightarrow} \infty. \end{aligned}$$

Hence, if $a = 1$, $|F'|$ is not bounded and F is thus not of linear growth. Instead, $F'(\xi) \simeq \log(1+|\xi|)$, indicating that $a = 1$ corresponds to $L\log L$-growth behaviour of F. This is an important distinction, since for linear growth functionals substantially worse compactness properties are available than for those of $L\log L$-growth.

As a consequence, integrands satisfying (2.76) exhibit a gap in the growth exponent a appearing on the left-hand side and 1 on the right-hand side. Thus, they resemble (p,q)-type problems *on the level of second derivatives.* For comparison, the vastly studied class of usual (p,q)-type problems consists of integrands $G\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying e.g. the following set of assumptions:

$$\begin{cases} c_1|z|^p - c_2 \leqslant G(z) \leqslant c_3(1+|z|^q), \\ c_4(1+|\xi|^2)^{-p/2}|z|^2 \leqslant \langle G''(\xi)z,z\rangle \leqslant c_5(1+|\xi|^2)^{-q/2}|z|^2 \end{cases} \qquad \text{for all } z,\xi \in \mathbb{R}^{N\times n}. \tag{2.77}$$

Integrands G satisfying (2.77) hence usually show different growth behaviour from above and below *both on the level of the function and the second derivatives.* As

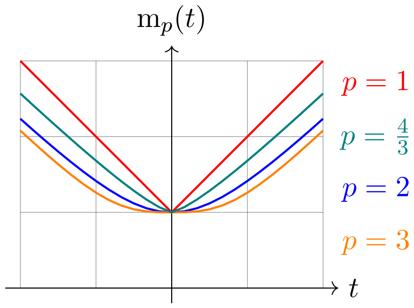

FIGURE 3. The scale $(\mathrm{m}_p)_{1\leqslant p<\infty}$. For $p > 2$, the graph of m_p is very flat at zero, thus leading to degeneracy of its second derivatives, $\mathrm{m}_p''(0) = 0$, whereas the second derivatives become singular for $|t| \searrow 0$ if $1 < p < \infty$. In the borderline case $p = 1$, ellipticity is completely lost also away from zero.

opposed to this, integrands of linear growth *typically have the same growth behaviour from above and below* whereas they have different growth behaviour from above and below *on the level of the second derivatives* only. This property is inherent to the very borderline case to be studied here, making it even a bit more involved to apply (p, q)-techniques.

We shall now discuss some examples that fall into the realm of a-ellipticity regimes.

Example 2.60. *Let* $1 < a < \infty$. *We define* $\widetilde{\Phi}_a\colon \mathbb{R} \to \mathbb{R}$ *by*

$$\widetilde{\Phi}_a(t) := \int_0^t \int_0^s \frac{\mathrm{d}\theta\,\mathrm{d}s}{(1+|\theta|^2)^{a/2}}, \qquad t \in \mathbb{R}.$$

Then, setting $\Phi_a(z) := \widetilde{\varphi}_a(|z|)$ *for* $z \in X$, *produces an* a*-elliptic integrand* $\varphi_a\colon Z \to \mathbb{R}$.

The foremost example to keep in mind is that of the area-integrand which we classify within another scale:

Example 2.61. *Given* $1 < p < \infty$, *we define* $\mathrm{m}_p\colon X \to \mathbb{R}$ *by*

$$\mathrm{m}_p(z) := \big(1 + |z|^p\big)^{\frac{1}{p}}, \qquad z \in X.$$

The usual area-integrand then is retrieved by setting $p = 2$ *as can be seen from the estimate*

$$\frac{c_1}{\sqrt{1+|\xi|^2}}\Big(|z|^2 - \frac{\langle \xi, z\rangle}{1+|\xi|^2}\Big) \leqslant \langle \mathrm{m}_2''(\xi)z, z\rangle \leqslant \frac{c_2}{\sqrt{1+|\xi|^2}}\Big(|z|^2 - \frac{\langle \xi, z\rangle}{1+|\xi|^2}\Big)$$

for all $\xi, z \in X$ *and two independent constants* $c_1, c_2 > 0$. *Within the scale of convex integrands* $(\mathrm{m}_p)_{1\leqslant p<\infty}$, *the area integrand* m_2 *is the only member to be* a*–elliptic for some suitable* a. *Indeed, as pointed out by* SCHMIDT [65, p. 7], *if*[7] $p > 1$, *then there exists* $C_p > 0$ *such that*

$$\begin{aligned} C_p^{-1}|\xi|^{p-2}|z|^2 &\leqslant \langle \mathrm{m}_p''(\xi)z, z\rangle \leqslant C_p|\xi|^{p-2}|z|^2 && \text{if } |\xi| \leqslant 1,\\ C_p^{-1}|\xi|^{-1-p}|z|^2 &\leqslant \langle \mathrm{m}_p''(\xi)z, z\rangle \leqslant C_p|\xi|^{-1}z|^2 && \text{if } |\xi| > 1. \end{aligned}$$

[7]The case $p = 1$ is excluded since $\mathrm{m}_1(\xi) - 1 = |\xi|$ which is a positively 1–homogeneous function and hence not even strictly convex.

In particular, if $p > 2$, then m_p'' vanishes at $\xi = 0$ (and so degenerates) whereas if $1 < p < 2$, then $\mathrm{m}_p''(\xi)$ becomes singular as $|\xi| \searrow 0$. This is illustrated in Figure 2.6.

2.7. **Viscosity approximations:** $1 < a < 1 + \frac{2}{n}$. We now set up the Ekeland-type viscosity approximation scheme, and hereafter suppose that $f \in \mathrm{C}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is convex with linear growth and $u_0 \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$. For ease of notation, we write $\overline{F} := \overline{\mathscr{F}}_{u_0}[-;\Omega]$ in the sequel. Let $u \in \mathrm{GM}(\mathscr{F};u_0)$ be arbitrary. By smooth approximation in the area-strict topology we find a sequence $(u_j) \subset \mathscr{D}_{u_0} := u_0 + \mathrm{W}^{1,1}_0(\Omega;\mathbb{R}^N)$ such that

$$u_j \to u \text{ area-strictly in } \mathrm{BV}(\Omega;\mathbb{R}^N). \tag{2.78}$$

Since $\overline{F}$ is continuous for the area-strict topology, (u_j) is a minimising sequence for $\overline{F}$, and we have $\mathscr{F}[u_j] = \overline{F}[u_j] \to \overline{F}[u] = \min \overline{F}[\mathrm{BV}(\Omega;\mathbb{R}^N)]$. Passing to a non-relabeled subsequence, we may thus assume

$$\min \overline{\overline{F}}[\mathrm{BV}(\Omega;\mathbb{R}^N)] \leqslant \mathscr{F}[u_j] \leqslant (\min \overline{\mathscr{F}}[\mathrm{BV}(\Omega;\mathbb{R}^N)]) + \frac{1}{8j^2} \qquad \text{for all } j \in \mathbb{N}. \tag{2.79}$$

Since the trace operator $\mathrm{Tr}\colon \mathrm{W}^{1,1}(\mathbb{R}^n \setminus \overline{\Omega};\mathbb{R}^N) \to \mathrm{L}^1(\partial\Omega;\mathbb{R}^N)$ is surjective, we find a compactly supported extension $\overline{u}_0 \in \mathrm{W}^{1,1}_c(\mathbb{R}^n;\mathbb{R}^N)$ of u_0. After a routine mollification of $\overline{u}_0$, we obtain $u_j^{\partial\Omega} \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)$ such that

$$\|u_j^{\partial\Omega} - u_0\|_{\mathrm{W}^{1,1}(\Omega)} \leqslant \frac{1}{8\,\mathrm{Lip}(f)j^2}, \tag{2.80}$$

where $\mathrm{Lip}(f)$ is the Lipschitz constant of f. We then put $\mathscr{D}_j := u_j^{\partial\Omega} + \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n) \subset \mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)$. Since $u_j - u_0 \in \mathrm{W}^{1,1}_0(\Omega;\mathbb{R}^N)$, we find $\widetilde{u}_j \in \mathscr{D}_j$ such that

$$\|u_j - u_0 - (\widetilde{u}_j - u_j^{\partial\Omega})\|_{\mathrm{W}^{1,1}(\Omega)} \leqslant \frac{1}{8\,\mathrm{Lip}(f)j^2},$$

from where it follows that

$$\|\widetilde{u}_j - u_j\|_{\mathrm{W}^{1,1}(\Omega)} \leqslant \|u_j - u_0 - (\widetilde{u}_j - u_j^{\partial\Omega})\|_{\mathrm{W}^{1,1}(\Omega)} + \|u_0 - u_j^{\partial\Omega}\|_{\mathrm{W}^{1,1}(\Omega)} \leqslant \frac{1}{4\,\mathrm{Lip}(f)j^2}. \tag{2.81}$$

Since $\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^N) \subset \mathrm{W}^{1,1}_0(\Omega;\mathbb{R}^N)$, we find for arbitrary $\varphi \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^N)$:

$$\begin{aligned}
\inf F[\mathscr{D}_{u_0}] &\leqslant F[u_0+\varphi] \\
&= F[u_0+\varphi] - F[u_j^{\partial\Omega}+\varphi] + F[u_j^{\partial\Omega}+\varphi] \\
&\leqslant \mathrm{Lip}(f)\|D(u_0 - u_j^{\partial\Omega})\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})} + F[u_j^{\partial\Omega}+\varphi] \\
&\overset{(2.80)}{\leqslant} \frac{1}{8j^2} + F[u_j^{\partial\Omega}+\varphi].
\end{aligned}$$

At this stage, we infimise the previous overall inequality over all $\varphi \in \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^n)$ to obtain

$$\inf F[\mathscr{D}_{u_0}] \leqslant \frac{1}{8j^2} + \inf F[\mathscr{D}_j]. \tag{2.82}$$

Then, since $\min \overline{F}[\mathrm{BV}(\Omega;\mathbb{R}^N)] = \inf F[\mathscr{D}_{u_0}]$, we deduce that

$$
\begin{aligned}
F[\widetilde{u}_j] &\leqslant F[\widetilde{u}_j] - F[u_j] + F[u_j] \\
&\leqslant \mathrm{Lip}(f)\|\varepsilon(\widetilde{u}_j) - \varepsilon(u_j)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})} + F[u_j] \\
&\overset{(2.81)}{\leqslant} \frac{1}{4j^2} + F[u_j] \overset{(2.79)}{\leqslant} \frac{3}{8j^2} + \inf F[\mathscr{D}_{u_0}] \\
&\overset{(2.82)}{\leqslant} \frac{1}{2j^2} + \inf F[\mathscr{D}_j].
\end{aligned} \tag{2.83}
$$

We consequently introduce the quantities A_j and the integrands $f_j\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ via

$$
A_j := 1 + \int_\Omega (1 + |D\widetilde{u}_j|^2)\,\mathrm{d}x \quad \text{and} \quad f_j(\xi) := f(\xi) + \frac{1}{2A_j j^2}(1 + |\xi|^2) \tag{2.84}
$$

for $\xi \in \mathbb{R}^{N\times n}$. In order to employ the Ekeland variational principle with respect to sufficiently weak perturbations, we extend the integral functionals corresponding to f_j to $\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)$ by

$$
F_j[w] := \begin{cases} \displaystyle\int_\Omega f_j(Dw)\,\mathrm{d}x & \text{if } w \in \mathscr{D}_j, \\ +\infty & \text{if } w \in \mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N) \setminus \mathscr{D}_j. \end{cases} \tag{2.85}
$$

For each $j \in \mathbb{N}$, the functional F_j is not identically $+\infty$ on $\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N)$. The latter space is Banach by Lemma 2.58 (a) and, by Lemma 2.59 with $\mathfrak{f} = f_j$, $q = 2$ and $k = 2$, F_j is lower semicontinuous with respect to the norm topology on $\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)$. Moreover, we record

$$
F_j[\widetilde{u}_j] \leqslant F[\widetilde{u}_j] + \frac{1}{2j^2} \overset{(2.83)}{\leqslant} \frac{1}{j^2} + \inf F[\mathscr{D}_j] \leqslant \frac{1}{j^2} + \inf F_j[\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)],
$$

having used the very definition of F_j in the ultimate step. Therefore, Ekeland's variational principle provides us with $v_j \in \mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N)$ such that

$$
\begin{aligned}
&\|v_j - \widetilde{u}_j\|_{\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N)} \leqslant \frac{1}{j}, \\
&F_j[v_j] \leqslant F_j[w] + \frac{1}{j}\|v_j - w\|_{\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N)} \qquad \text{for all } w \in \mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N).
\end{aligned} \tag{2.86}
$$

We extract from (2.86) some routine information by testing with $w = \widetilde{u}_j$:

$$
\begin{aligned}
F_j[v_j] &\overset{(2.86)_2}{\leqslant} F_j[\widetilde{u}_j] + \frac{1}{j}\|v_j - \widetilde{u}_j\|_{\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)} \\
&\overset{(2.86)_1}{\leqslant} F[\widetilde{u}_j] + \frac{1}{2A_j j^2}\int_\Omega (1 + |\varepsilon(\widetilde{u}_j)|^2)\,\mathrm{d}x + \frac{1}{j^2} \\
&\overset{(2.83)}{\leqslant} \inf F[\mathscr{D}_{u_0}] + \frac{2}{j^2}.
\end{aligned} \tag{2.87}
$$

The latter quantity is finite and so, by the very definition of F_j, $v_j \in \mathscr{D}_j \subset \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$. Moreover, as $v_j - u_j^{\partial\Omega} \in \mathrm{W}_0^{1,2}(\Omega;\mathbb{R}^N) \subset \mathrm{W}_0^{1,1}(\Omega;\mathbb{R}^N)$,

$$\begin{aligned}\inf F[\mathscr{D}_{u_0}] &\leqslant F[u_0 + (v_j - u_j^{\partial\Omega})] - F[v_j] + F[v_j] \\ &\leqslant \mathrm{Lip}(f)\|u_0 - u_j^{\partial\Omega}\|_{\mathrm{LD}(\Omega)} + F[v_j] \\ &\overset{(2.80)}{\leqslant} \frac{1}{8j^2} + F_j[v_j] \overset{(2.87)}{\leqslant} \frac{3}{j^2} + \inf F[\mathscr{D}_{u_0}].\end{aligned} \tag{2.88}$$

For latter purposes, we record the *perturbed Euler-Lagrange equation*

$$\left|\int_\Omega \langle f_j'(Dv_j), D\varphi\rangle\,\mathrm{d}x\right| \leqslant \frac{1}{j}\|\varphi\|_{\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N)} \qquad \text{for all } \varphi \in \mathrm{W}_c^{1,2}(\Omega;\mathbb{R}^N). \tag{2.89}$$

This inequality can be obtained by testing $(2.86)_2$ with $w = v_j \pm \theta\varphi$ for $\theta > 0$, $\varphi \in \mathrm{W}_c^{1,2}(\Omega;\mathbb{R}^n)$, dividing the resulting inequalities by θ and then sending $\theta \searrow 0$. Moreover, by the linear growth hypothesis and $c = \min\{\frac{1}{2}, c_1\}$, we infer from (2.87) that

$$\int_\Omega |Dv_j|\,\mathrm{d}x + \frac{1}{A_j j^2}\int_\Omega (1 + |Dv_j|^2)\,\mathrm{d}x \leqslant c\Big(\inf F[\mathscr{D}_{u_0}] + \gamma\mathscr{L}^n(\Omega) + \frac{2}{j^2}\Big) \tag{2.90}$$

holds for all $j \in \mathbb{N}$. Finally, we note that due to Poincaré's inequality on $\mathrm{W}_0^{1,1}(\Omega;\mathbb{R}^N)$ we obtain

$$\begin{aligned}\sup_{j\in\mathbb{N}} \int_\Omega |v_j|\,\mathrm{d}x &\leqslant \sup_{j\in\mathbb{N}}\Big[\int_\Omega |v_j - u_j^{\partial\Omega}|\,\mathrm{d}x + \int_\Omega |u_j^{\partial\Omega}|\,\mathrm{d}x\Big] \\ &\leqslant 2C\sup_{j\in\mathbb{N}}\Big[\int_\Omega |Dv_j|\,\mathrm{d}x + \|u_j^{\partial\Omega}\|_{\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)}\Big] \overset{(2.80),\,(2.90)}{<} \infty,\end{aligned} \tag{2.91}$$

where $C > 0$ is the constant appearing in the requisite Poincaré inequality. We finally record

Lemma 2.62. *The sequence (v_j) as constructed in* (2.86) *possesses a subsequence $(v_{j(l)}) \subset (v_j)$ such that*

$$v_{j(l)} \overset{*}{\rightharpoonup} u \qquad \text{in } \mathrm{BV}(\Omega;\mathbb{R}^N) \text{ as } l \to \infty,$$

where $u \in \mathrm{GM}(\mathscr{F};u_0)$ is the generalised minimiser fixed in the beginning of the section.

Proof. By (2.90) and (2.91) we conclude that (v_j) is uniformly bounded in $\mathrm{BV}(\Omega;\mathbb{R}^N)$, and thus possesses a subsequence $(v_{j(l)}) \subset (v_j)$ such that $v_{j(l)} \overset{*}{\rightharpoonup} v$ in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ as $l \to \infty$ for some $v \in \mathrm{BD}(\Omega)$. Since $\mathrm{L}^1(\Omega;\mathbb{R}^N) \hookrightarrow \mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N)$ by Lemma 2.58(b), $v_{j(l)} \to v$ in $\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N)$. On the other hand, (2.78), (2.81) and (2.86) imply that $v_{j(l)} \to u$ in $\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N)$. and since $\mathrm{L}^1(\Omega;\mathbb{R}^N) \hookrightarrow \mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N)$ by Lemma 2.58 (b), and hence $u = v$. The proof is complete. ☐

2.7.1. *Preliminary regularity estimates.* To justify the manipulations on the perturbed Euler-Lagrange equations satisfied by the v_j's, we now derive *non-uniform* regularity estimates. Since (2.89) do not display elliptic differential equations (but differential inequalities), the corresponding higher differentiability assertions need to be approached slightly more carefully than for plain viscosity methods:

Lemma 2.63. *Let $f \in \mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be of linear growth and, for some $\Lambda \in (0,\infty)$, satisfy the bound*

$$0 < \langle f''(z)\xi,\xi\rangle \leqslant \Lambda \frac{|\xi|^2}{(1+|z|^2)^{\frac{1}{2}}} \qquad \text{for all } z,\xi \in \mathbb{R}^{N\times n}. \tag{2.92}$$

Define v_j for $j\in\mathbb{N}$ by (2.86). Then there holds $v_j \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$.

Proof. Let $x_0\in\Omega$ and $0<r<R<\operatorname{dist}(x_0,\partial\Omega)$. Also, let $s\in\{1,...,n\}$, $0<h<\frac{1}{2}\operatorname{dist}(x_0,\partial\Omega)-R$ and pick $\rho\in\mathrm{C}^\infty_c(\Omega;[0,1])$ be such that $\mathbb{1}_{\mathrm{B}(x_0,r)}\leqslant\rho\leqslant\mathbb{1}_{\mathrm{B}(x_0,R)}$. We test the perturbed Euler-Lagrange equation (2.89) with $\varphi := \Delta_{s,-h}(\rho^2\Delta_{s,h}v_j) \in \mathrm{W}^{1,2}_c(\Omega;\mathbb{R}^N)$. In consequence, integration by parts for difference quotients yields

$$\left|\int_\Omega \langle \Delta_{s,h}f_j'(Dv_j), D(\rho^2\Delta_{s,h}v_j)\rangle\,\mathrm{d}x\right| \leqslant \frac{1}{j}\|\Delta_{s,-h}(\rho^2\Delta_{s,h}v_j)\|_{\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N)}. \tag{2.93}$$

We define for $\mathscr{L}^n$-a.e. $x\in\mathrm{B}(x_0,R)$ bilinear forms $\mathscr{B}_{j,s,h}(x)\colon \mathbb{R}^{N\times n}\times\mathbb{R}^{N\times n}\to\mathbb{R}$ by

$$\mathscr{B}_{j,s,h}(x)[\eta,\xi] := \int_0^1 \langle f_j''(\varepsilon(v_j)(x)+th\Delta_{s,h}\varepsilon(v_j)(x))\eta,\xi\rangle\,\mathrm{d}t, \qquad \xi,\eta\in\mathbb{R}^{N\times n}.$$

Then we note that, because of (2.92) and the definition of f_j,

$$(j^2A_j)^{-1}|\xi|^2 \leqslant \mathscr{B}_{j,s,h}(x)[\xi,\xi] \leqslant (\Lambda+(j^2A_j)^{-1})|\xi|^2 =: C_j|\xi|^2 \tag{2.94}$$

for all $\xi\in\mathbb{R}^{N\times n}$, independently from s,h and x. Thus each $\mathscr{B}_{j,s,h}(x)$ is an elliptic bilinear form itself and a suitable version of Young's inequality is available. With this notation, we infer from (2.93) by expanding the terms on the left and regrouping

$$\begin{aligned}
\mathbf{I} := \int_\Omega \mathscr{B}_{j,s,h}(x)[\rho D(\Delta_{s,h}v_j),\rho D(\Delta_{s,h}v_j)]\,\mathrm{d}x &\leqslant \int_\Omega \mathscr{B}_{j,s,h}(x)[\rho D(\Delta_{s,h}v_j), 2\nabla\rho\otimes\Delta_{s,h}v_j]\,\mathrm{d}x \\
&\quad + \frac{1}{j}\|\Delta_{s,-h}(\rho^2\Delta_{s,h}v_j)\|_{\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)} \\
&\leqslant \frac{1}{2}\int_\Omega \mathscr{B}_{j,s,h}(x)[\rho D(\Delta_{s,h}v_j),\rho D(\Delta_{s,h}v_j)]\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega \mathscr{B}_{j,s,h}(x)[2\nabla\rho\odot\Delta_{s,h}v_j, 2\nabla\rho\odot\Delta_{s,h}v_j]\,\mathrm{d}x \\
&\quad + \frac{1}{j}\|\Delta_{s,-h}(\rho^2\Delta_{s,h}v_j)\|_{\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N)} =: \mathbf{II}+\mathbf{III}+\mathbf{IV}.
\end{aligned}$$

Absorbing term **II** into **I**, we obtain

$$\frac{1}{2j^2A_j}\int_\Omega |\rho D(\Delta_{s,h}v_j)|^2\,\mathrm{d}x \overset{(2.94)}{\leqslant} \frac{1}{2}\mathbf{I} = \mathbf{I}-\mathbf{II} \leqslant \mathbf{III}+\mathbf{IV} \tag{2.95}$$

and thus need to give bounds on **III** and **IV**. As a consequence of (2.94), we immediately obtain

$$\mathbf{III} \leqslant 4C_j\sup_\Omega|\nabla\rho|^2\int_{\mathrm{B}(x_0,R)}|\Delta_{s,h}v_j|^2\,\mathrm{d}x \leqslant 4C_j(\sup_\Omega|\nabla\rho|^2)\|v_j\|^2_{\mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)}$$

which is finite due to $v_j\in\mathrm{W}^{1,2}(\Omega;\mathbb{R}^n)$. Moreover, $\mathbf{IV}\leqslant cj^{-1}\|v_j\|_{\mathrm{L}^1}$. The second term on the very right hand side of inequality (**??**) consequently is absorbed into the very left hand side of (2.95), and then we obtain $\sup_{|h|<\operatorname{dist}(x_0,\partial\Omega)-R}\mathbf{I}<\infty$. Thus, $(\Delta_{s,h}\,\mathrm{D}v_j)_h$ is uniformly bounded in $\mathrm{L}^2(\mathrm{B}(x_0,r);\mathbb{R}^{N\times n})$ and hence $\partial_s\varepsilon(v_j)$

exists in $\mathrm{L}^2(\mathrm{B}(x_0,r);\mathbb{R}^{N\times n})$ for each $s\in\{1,...,n\}$. As a consequence, $\partial_s v_j\in \mathrm{W}^{1,2}(\mathrm{B}(x_0,r);\mathbb{R}^N)$. By arbitrariness of $s\in\{1,...,n\}$, $x_0\in\Omega$ and $R>0$ sufficiently small, we thus obtain $v_j\in\mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. The proof is complete. □

2.7.2. *Uniform second order estimates.* Using the $\mathrm{W}^{2,2}_{\mathrm{loc}}$-regularity of the Ekeland viscosity sequence (v_j), we now establish *uniform* second order estimates as follows:

Proposition 2.64. *There exists a constant $c>0$ such that for the sequence (v_j) as constructed above there holds for any $\omega\Subset\Omega$:*

$$\sup_{j\in\mathbb{N}}\int_\omega \frac{|D^2v_j|^2}{(1+|Dv_j|^2)^{\frac{a}{2}}}\,\mathrm{d}x =: C_\omega<\infty. \tag{2.96}$$

In the following, it is customary to introduce the shorthand notation

$$\sigma_j := f_j'(Dv_j)\quad\text{and}\quad \mathcal{A}_j[\nu;\xi,\eta]:=\langle f_j''(\nu)\xi,\eta\rangle,\qquad \nu,\xi,\eta\in\mathbb{R}^{N\times n},$$

Lemma 2.65. *Let the integrand $f\in\mathrm{C}^2(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ satisfy the requirements of the previous lemma. Define v_j by (2.86). Then for all $\ell\in\{1,...,n\}$ and $\varphi\in\mathrm{W}^{1,2}_c(\Omega;\mathbb{R}^n)$ there holds*

$$\left|\int_\Omega\langle\partial_\ell\sigma_j,D\varphi\rangle\,\mathrm{d}x\right|\leqslant\frac{1}{j}\|\varphi\|_{\mathrm{W}^{-1,1}(\Omega;\mathbb{R}^n)}. \tag{2.97}$$

Proof. By Lemma 2.63, $v_j\in\mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$. We note that $\partial_\ell\sigma_j=f_j''(Dv_j)\partial_\ell Dv_j$, and since $\sup_{z\in\mathbb{R}^{N\times n}}|f_j''(z)|<\infty$, $\sigma_j\in\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$. Let $\varphi\in\mathrm{C}^\infty_c(\Omega;\mathbb{R}^N)$. Then $\partial_\ell\varphi$ is an admissible competitor in (2.89) and so, since $\sigma_j\in\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$,

$$\left|\int_\Omega\langle\partial_\ell\sigma_j,D\varphi\rangle\,\mathrm{d}x\right|=\left|\int_\Omega\langle\sigma_j,D(\partial_\ell\varphi)\rangle\,\mathrm{d}x\right|\overset{(2.89)}{\leqslant}\frac{1}{j}\|\partial_\ell\varphi\|_{\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^N)}\leqslant\frac{1}{j}\|\varphi\|_{\mathrm{W}^{-1,1}(\Omega;\mathbb{R}^N)}.$$

Here, the last estimate is valid by Lemma 2.58 (b). Then the case of general $\mathrm{W}^{1,2}_c(\Omega;\mathbb{R}^N)$-maps φ follows by routine smooth approximation and $\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)\hookrightarrow\mathrm{W}^{-1,1}(\Omega;\mathbb{R}^N)$. □

Proof of Proposition 2.64. We test the Euler-Lagrange equation of the previous lemma with $\varphi=\rho^2\partial_\ell v_j$ for some localisation function ρ. Then we estimate

$$\begin{aligned}\int_\Omega\langle\partial_\ell\sigma_j,\rho^2\partial_\ell Dv_j\rangle\,\mathrm{d}x &\leqslant\frac{1}{j}\|\varphi\|_{\mathrm{W}^{-1,1}(\Omega;\mathbb{R}^N)}+\left|\int_\Omega\langle\partial_\ell\sigma_j,2\rho\nabla\rho\otimes\partial_\ell v_j\rangle\,\mathrm{d}x\right|\\ &\leqslant\frac{1}{j}\|\varphi\|_{\mathrm{W}^{-1,1}(\Omega;\mathbb{R}^N)}+\frac{1}{2}\int_\Omega\langle f_j''(Dv_j)\rho\partial_\ell Dv_j,\rho\partial_\ell Dv_j\rangle\,\mathrm{d}x\\ &\quad+\frac{1}{2}\int_\Omega\langle f_j''(Dv_j)(\partial_\ell v_j\otimes\nabla\varphi),(\partial_\ell v_j\otimes\nabla\varphi)\rangle\,\mathrm{d}x.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2}\int_\Omega\langle f_j''(Dv_j)\rho\partial_\ell Dv_j,\rho\partial_\ell Dv_j\rangle\,\mathrm{d}x&\leqslant\frac{1}{j}\|\varphi\|_{\mathrm{W}^{-1,1}(\Omega;\mathbb{R}^N)}\\ &\quad+\frac{1}{2}\int_\Omega\langle f_j''(Dv_j)(\partial_\ell v_j\otimes\nabla\varphi),(\partial_\ell v_j\otimes\nabla\varphi)\rangle\,\mathrm{d}x.\end{aligned}$$

Now,

$$\frac{1}{j}\|\varphi\|_{\mathrm{W}^{-1,1}}\leqslant\frac{1}{j}\|\partial_\ell(\rho^2v_j)-2\rho v_j\otimes\nabla\rho\|_{\mathrm{W}^{-1,1}}\leqslant\frac{c}{j}\|v_j\|_{\mathrm{L}^1}.$$

On the other hand,

$$\begin{aligned}\frac{1}{2}\int_{\Omega}\langle f_j''(Dv_j)(v_j\otimes\nabla\varphi),(v_j\otimes\nabla\varphi)\rangle\,\mathrm{d}x &\leqslant c\int_{\mathrm{spt}(\varphi)}\frac{|Dv_j|^2}{(1+|Dv_j|)}+\frac{1}{2A_jj^2}(1+|Dv_j|^2)\,\mathrm{d}x\\ &\leqslant c<\infty\end{aligned}$$

by the available a priori-estimates. Now the condition of a-ellipticity yields the claimed result. □

2.7.3. *Regularity in the regime* $1<a<1+\frac{2}{n}$. Working from the previous section, we can now turn to the first regularity for a-elliptic integrands.

Theorem 2.66. *Suppose that* $f\in\mathrm{C}^2(\mathbb{R}^{N\times n})$ *is an* a*-elliptic integrand of linear growth, where* $1<a<1+\frac{2}{n}$. *Then any generalised minimiser of* $\mathscr{F}$ *belongs to* $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$.

Proof. To be inserted. □

2.8. **Regularity in the regime** $1+\frac{2}{n}<a\leqslant 3$**.** Compared with the previous section, we now weaken the ellipticity beyond the dimensional threshold $1+\frac{2}{n}$. As is common in this situation, to obtain regularity results without dimensional restriction on the ellipticity a, we must augment our setting by *additional conditions on the minimisers*. In our setting, we suppose that $u\in\mathrm{BV}(\Omega;\mathbb{R}^N)$ is a generalised minimiser of an a-elliptic variational integral which, in addition, satisfies,

$$u\in\mathrm{L}^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^N). \tag{2.98}$$

Aiming for a $\mathrm{W}^{1,1}$-regularity result as in the previous section, we encounter the following *difficulty*: The Ekeland variational principle (employed in negative Sobolev spaces) does not immediately allow us to keep track of the corresponding local L^∞-bounds. Hence modification is required, and this section aims at explaining how such a modification can be established.

2.8.1. *Refined Ekeland approximation.* $1+\frac{2}{n}\leqslant a\leqslant 3$. Here our aim is to establish

$$\mathrm{GM}(\mathscr{F};u_0)\cap\mathrm{L}^\infty_{\mathrm{loc}}\subset\mathrm{W}^{1,L\log^2L}_{\mathrm{loc}}(\Omega;\mathbb{R}^N).$$

In this case, we let $u\in\mathrm{GM}(F;u_0)\cap\mathrm{L}^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$. Toward a $\mathrm{W}^{1,1}$-regularity result which, in turn, here is a local statement, we let $\mathrm{B}:=\mathrm{B}(x_0,R)\Subset\Omega$ be a ball. Then for any $\varepsilon>0$ such that $\mathrm{B}':=\mathrm{B}(x_0,R+\varepsilon)\Subset\Omega$ we have $u|_{\mathrm{B}(x_0,R+\varepsilon)}\in\mathrm{L}^\infty(\mathrm{B}(x_0,R+\varepsilon);\mathbb{R}^n)$. We note that generalised minima are local generalised minima in the sense that for any open Lipschitz subset $K\Subset\Omega$ there holds $\overline{F}_u[u;K]\leqslant\overline{F}_u[v;K]$ for all $v\in\mathrm{BV}(K;\mathbb{R}^N)$. Hence, if we can establish that all generalised minima v of $\overline{F}_u[-;\mathrm{B}']$ are of class $\mathrm{W}^{1,L\log L}_{\mathrm{loc}}(\mathrm{B}';\mathbb{R}^N)$ for $u\in\mathrm{L}^\infty(\mathrm{B}';\mathbb{R}^n)$, then we will equally obtain that all elements of $\mathrm{GM}(F;u_0)\cap\mathrm{L}^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ feature the $\mathrm{W}^{1,L\log L}_{\mathrm{loc}}$-regularity in Ω. Therefore we may directly assume that $u_0\in\mathrm{L}^\infty(\Omega;\mathbb{R}^n)$ and set $m:=\|u_0\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^n)}$.

Since we must keep track of the L^∞-constraint within the Ekeland approximation scheme, we proceed slightly more carefully than for case 1 and thus make use of a refined smooth approximation result. By Proposition **??**, there exists a number $M\in\mathbb{N}$ (depending on Ω only) and a sequence $(u_j)\subset\mathrm{L}^\infty_{\leqslant Mm}(\Omega;\mathbb{R}^n)\cap\mathscr{D}_{u_0}$ such that the convergence specified in (2.78) holds for (u_j). The existence of such a sequence must be established by different means than as explained in [19, Eq. (5.3)ff.] within the BV-framework, cf. Remark 2.68 and Appendix A below. Then we analogously infer (2.79).

Given some $R'>0$ such that $\Omega \Subset \mathrm{B}(0,R')$, we denote $\mathfrak{J}\colon \mathrm{LD}(\Omega)\to \mathrm{LD}_0(\mathrm{B}(0,R'))$ the JONES-type extension operator from Proposition ??. Then $\mathfrak{J}\colon \mathrm{LD}(\Omega)\cap \mathrm{L}^{\infty}(\Omega;\mathbb{R}^n)\to \mathrm{L}^{\infty}(\mathrm{B}(0,R');\mathbb{R}^n)$ is a bounded linear operator with respect to the L^{∞}-norm. We put

$$\|\mathfrak{J}\| := \sup\{\|\mathfrak{J}v\|_{\mathrm{L}^{\infty}(\mathrm{B}(0,R');\mathbb{R}^n)}\colon\; v\in \mathrm{LD}(\Omega)\cap \mathrm{L}^{\infty}(\Omega;\mathbb{R}^n),\; \|v\|_{\mathrm{L}^{\infty}(\Omega;\mathbb{R}^n)}\leqslant 1\}$$

and define $\overline{u}_0 := \mathfrak{J}u_0$. As usual, denote $\rho\in \mathrm{C}_c^{\infty}(\mathrm{B}(0,1);[0,1])$ a radial standard mollifier with $\|\rho\|_{\mathrm{L}^1(\mathrm{B}(0,1))}=1$ and put, for $\varepsilon>0$, $\rho_{\varepsilon}(x):=\varepsilon^{-n}\rho(\frac{x}{\varepsilon})$. For each $j\in\mathbb{N}$, we choose $\varepsilon_j>0$ such that $\overline{u}_j^{\partial\Omega}:=\rho_{\varepsilon_j}*\overline{u}_0$ and $u_j^{\partial\Omega}:=\overline{u}_j^{\partial\Omega}|_{\Omega}$ satisfy

$$\left\{\begin{array}{l} \|u_j^{\partial\Omega}-u_0\|_{\mathrm{LD}(\Omega)}\leqslant \|\overline{u}_j^{\partial\Omega}-\overline{u}_0\|_{\mathrm{LD}(\mathbb{R}^n)}\leqslant \frac{1}{8\,\mathrm{Lip}(f)j^2},\\ \|u_j^{\partial\Omega}\|_{\mathrm{L}^{\infty}(\Omega;\mathbb{R}^n)}\leqslant \|\mathfrak{J}\|\|u_0\|_{\mathrm{L}^{\infty}(\Omega;\mathbb{R}^n)}=\|\mathfrak{J}\|m\end{array}\right. \tag{2.99}$$

seriatim. Having constructed $(u_j^{\partial\Omega})\subset(\mathrm{W}^{1,n+1}\cap \mathrm{L}^{\infty}_{\leqslant\|\mathfrak{J}\|m})(\Omega;\mathbb{R}^n)$, we then define $\mathscr{D}_j:=u_j^{\partial\Omega}+\mathrm{W}_0^{1,n+1}(\Omega;\mathbb{R}^n)$. Again, $u_j-u_0\in\mathrm{LD}_0(\Omega;\mathbb{R}^n)$ and a usual mollification procedure provides us with $\psi_j\in \mathrm{C}_c^{\infty}(\Omega;\mathbb{R}^n)$ such that $\|\psi_j-(u_j-u_0)\|_{\mathrm{LD}(\Omega)}\leqslant \frac{1}{8\,\mathrm{Lip}(f)j^2}$ together with $\|\psi_j\|_{\mathrm{L}^{\infty}(\Omega;\mathbb{R}^n)}\leqslant\|u_j-u_0\|_{\mathrm{L}^{\infty}(\Omega;\mathbb{R}^n)}\leqslant(M+1)m$. Put $\widetilde{u}_j:=u_j^{\partial\Omega}+\psi_j\in\mathscr{D}_j$. Then we deduce that

$$\begin{aligned} &\|u_j-u_0-(\widetilde{u}_j-u_j^{\partial\Omega})\|_{\mathrm{LD}(\Omega)}=\|u_j-u_0-\psi_j\|_{\mathrm{LD}(\Omega)}\leqslant\frac{1}{8\,\mathrm{Lip}(f)j^2},\\ &\|\widetilde{u}_j\|_{\mathrm{L}^{\infty}(\Omega;\mathbb{R}^n)}\leqslant\|u_j^{\partial\Omega}\|_{\mathrm{L}^{\infty}(\Omega;\mathbb{R}^n)}+\|\psi_j\|_{\mathrm{L}^{\infty}(\Omega;\mathbb{R}^n)}\leqslant(1+M+\|\mathfrak{J}\|)m\end{aligned} \tag{2.100}$$

hold for all $j\in\mathbb{N}$. For notational brevity in the following, we put

$$\Gamma:=(1+M+\|\mathfrak{J}\|)m.$$

In addition, examining the construction of $\widetilde{u}_j$, we have

$$\|\widetilde{u}_j\|_{\mathrm{W}^{1,n+1}(\Omega;\mathbb{R}^n)}\leqslant\Upsilon(j)\|u_0\|_{\mathrm{LD}(\Omega)} \tag{2.101}$$

with $\Upsilon\colon\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ being a convex and strictly monotonously increasing function with $\Upsilon(t)\to\infty$ as $t\to\infty$; note that Υ only depends on Ω and f. Then estimate (2.81) is available in this setting, too. As a consequence, (2.82) and (2.83) hold for the sequences $(\widetilde{u}_j)$ and (u_j), too.

We again aim to apply the Ekeland variational principle, but this time *need to keep control of the* L^{∞}*-bounds*. To do so, we now modify the viscosity approximation scheme from above and continue by letting $C_{\mathtt{opt}}>0$ be the optimal constant for the Morrey embedding $\mathrm{W}^{1,n+1}(\Omega;\mathbb{R}^n)\hookrightarrow \mathrm{C}^{0,\frac{1}{n+1}}(\overline{\Omega};\mathbb{R}^n)$ so that, in particular,

$$|\Theta(x)-\Theta(y)|\leqslant C_{\mathtt{opt}}\|\Theta\|_{\mathrm{W}^{1,n+1}(\Omega;\mathbb{R}^n)}|x-y|^{\frac{1}{n+1}}\qquad\text{for all }\Theta\in\mathrm{W}^{1,n+1}(\Omega;\mathbb{R}^n). \tag{2.102}$$

Equally, we let $C_{\mathtt{Korn},n}>0$ be the optimal constant such that the Korn-type inequality

$$\|\Theta\|_{\mathrm{W}^{1,n+1}(\Omega;\mathbb{R}^n)}\leqslant C_{\mathtt{Korn},n}(\|\Theta\|_{\mathrm{L}^{n+1}(\Omega;\mathbb{R}^n)}+\|\varepsilon(\Theta)\|_{\mathrm{L}^{n+1}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}) \tag{2.103}$$

holds for all $\Theta \in \mathrm{W}^{1,n+1}(\Omega;\mathbb{R}^n)$. Next consider a function $h\colon [1,2)\to[0,\infty)$ which is continuous, convex, satisfies $h(1)=0$ and, for $\frac{3}{2}\leqslant t<2$,

$$\frac{1}{\omega_n}\Big(\frac{4C_{\mathtt{opt}}}{m}\Big(1+2^{\frac{n-1}{2}}(\mathscr{L}^n(\Omega)+(\Upsilon\Big(\frac{1}{2-|t|}\Big)\|u_0\|_{\mathrm{LD}(\Omega)})^{n+1})\Big)\frac{1}{(2-|t|)^4}\Big)^{n(n+1)}<h(t). \tag{2.104}$$

Since Υ can safely be chosen as a suitable multiple of the convex function $t\mapsto 1+\exp(t)$, such a function h is easily seen to exist.

We then consider the function $g\colon \mathbb{R}^{n\times n}_{\mathrm{sym}}\to\mathbb{R}\cup\{+\infty\}$ given by $g(\cdot)=\widetilde{g}(|\cdot|)$, where

$$\widetilde{g}(t):=\begin{cases}0 & \text{if } 0\leqslant t\leqslant 1,\\ h(t) & \text{if } 1\leqslant t<2,\\ +\infty & \text{if } t\geqslant 2,\end{cases} \tag{2.105}$$

and we record that the function g is convex, lower semicontinuous and its restriction to $\mathbb{B}(0,2)$ is of class C^2. We now put, slightly different from (2.84),

$$A_j:=1+\int_\Omega(1+|\varepsilon(\widetilde{u}_j)|^2)^{\frac{n+1}{2}}\,\mathrm{d}x\quad\text{and}\quad f_j(\xi):=f(\xi)+\frac{1}{2A_jj^2}(1+|\xi|^2)^{\frac{n+1}{2}} \tag{2.106}$$

for $\xi\in\mathbb{R}^{n\times n}_{\mathrm{sym}}$. Before we turn to the perturbed functional analogous to (2.85), let us note that if $j\geqslant 2$, then we have by a simple convexity estimate and (2.101)

$$A_j\leqslant 1+2^{\frac{n-1}{2}}(\mathscr{L}^n(\Omega)+\Upsilon(j)^{n+1}\|u_0\|^{n+1}_{\mathrm{LD}(\Omega)}).$$

Thus, choosing $t=2-\frac{1}{j}$ in (2.104), we find that

$$\frac{1}{\omega_n}\Big(\frac{4C_{\mathtt{opt}}A_jj^4}{m}\Big)^{n(n+1)}<\widetilde{g}\Big(2-\frac{1}{j}\Big), \tag{2.107}$$

an estimate that shall turn out important in the sequel. The substitute of (2.85) then is

$$F_j[w]:=\begin{cases}\displaystyle\int_\Omega f_j(\varepsilon(w))\,\mathrm{d}x+\int_\Omega g\Big(\frac{w}{\Gamma}\Big)\,\mathrm{d}x & \text{if } w\in\mathscr{D}_j,\\ +\infty & \text{if } w\in\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)\setminus\mathscr{D}_j.\end{cases} \tag{2.108}$$

It is then equally seen that F_j is lower semicontinuous with respect to the norm topology on $\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)$, cf. Lemma 2.59 with $\mathfrak{f}_1=f_j$, $\mathfrak{f}_2=g(\cdot/\Gamma)$, $q=n+1$ and $k=2$. Obviously, $F_j\not\equiv\infty$ on $\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)$. By $(2.100)_2$, the definition of g and the relevant analogue of (2.83),

$$F_j[\widetilde{u}_j]\leqslant F[\widetilde{u}_j]+\frac{1}{2j^2}\leqslant\frac{1}{j^2}+\inf F[\mathscr{D}_j]\leqslant\frac{1}{j^2}+\inf F_j[\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)]. \tag{2.109}$$

For each $j\in\mathbb{N}$, the Ekeland variational principle then yields $v_j\in\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)$ such that

$$\begin{aligned}&\|v_j-\widetilde{u}_j\|_{\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)}\leqslant\frac{1}{j},\\ &F_j[v_j]\leqslant F_j[w]+\frac{1}{j}\|v_j-w\|_{\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)}\qquad\text{for all } w\in\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n).\end{aligned} \tag{2.110}$$

Testing $(2.110)_2$ with $w = \widetilde{u}_j$ and using $(2.110)_1$ in conjunction with the analogue of (2.83) then yields

$$F_j[v_j] \leqslant \frac{2}{j^2} + \inf F[\mathscr{D}_{u_0}], \tag{2.111}$$

from where we deduce that $v_j \in \mathscr{D}_j$. Since $v_j \in \mathrm{C}^{0,\frac{1}{n+1}}(\overline{\Omega};\mathbb{R}^n)$, this readily gives $|v_j| \leqslant 2\Gamma$ and $\mathscr{L}^n(\{|v_j| = 2\Gamma\}) = 0$. This in itself, however, is *not good enough for the future derivation of the perturbed Euler-Lagrange equation*, cf. Remark 2.67. Thus we next address the issue of finding $j_0 \in \mathbb{N}$ such that $\sup_K |v_j| < 2\Gamma$ holds for all $j \geqslant j_0$ and any relatively compact subset K of Ω. Toward this aim, we utilise $v_j - u_j^{\partial\Omega} \in \mathrm{W}_0^{1,n+1}(\Omega;\mathbb{R}^n) \subset \mathrm{LD}_0(\Omega)$ and estimate

$$\begin{aligned}\inf F[\mathscr{D}_{u_0}] &\leqslant F[u_0 + (v_j - u_j^{\partial\Omega})] - F[v_j] + F[v_j]\\ &\leqslant \mathrm{Lip}(f)\|u_0 - u_j^{\partial\Omega}\|_{\mathrm{LD}(\Omega)} + F[v_j] \overset{(2.99)_1}{\leqslant} \frac{1}{8j^2} + F_j[v_j] \overset{(2.111)}{\leqslant} \frac{3}{j^2} + \inf F[\mathscr{D}_{u_0}].\end{aligned}$$

Sending $j \to \infty$ in the ultimate inequality, we find that $\lim_{j\to\infty} F_j[v_j] = \lim_{j\to\infty} F[v_j] = \inf F[\mathscr{D}_{u_0}]$. By the very definition of F_j, cf. (2.109), this readily implies $\lim_{j\to\infty} \mathbf{J}_j = 0$, where

$$\mathbf{J}_j := \frac{1}{2A_j j^2}\Big(\int_\Omega (1 + |\varepsilon(v_j)|^2)^{\frac{n+1}{2}}\,\mathrm{d}x\Big)^{\frac{1}{n+1}} + \int_\Omega g\Big(\frac{v_j}{\Gamma}\Big)\,\mathrm{d}x. \tag{2.112}$$

Working from here, we find $j_0 \in \mathbb{N}_{\geqslant 2}$ such that $j \geqslant j_0 \Rightarrow \mathbf{J}_j \leqslant 1/C_{\mathtt{Korn},n}$ and, moreover, $2(1+M+\|\mathfrak{I}\|)C_{\mathtt{Korn},n} m\mathscr{L}^n(\Omega)^{\frac{1}{n+1}} \leqslant 2A_j j^2$. We next estimate $\|v_j\|_{\mathrm{W}^{1,n+1}(\Omega;\mathbb{R}^n)}$ by virtue of

$$\begin{aligned}\|v_j\|_{\mathrm{W}^{1,n+1}(\Omega;\mathbb{R}^n)} &\leqslant C_{\mathtt{Korn},n}(\|v_j\|_{\mathrm{L}^{n+1}(\Omega;\mathbb{R}^n)} + \|\varepsilon(v_j)\|_{\mathrm{L}^{n+1}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})})\\ &\leqslant C_{\mathtt{Korn},n}\Big(2(1+M+\|\mathfrak{I}\|)m\mathscr{L}^n(\Omega)^{\frac{1}{n+1}} + \frac{2A_j j^2}{C_{\mathtt{Korn},n}}\Big) \leqslant 4A_j j^2.\end{aligned}$$

Now, as a consequence of (2.102), we find

$$|v_j(x) - v_j(y)| \leqslant 4C_{\mathtt{opt}} A_j j^2 |x-y|^{\frac{1}{n+1}} \qquad \text{for all } x, y \in \Omega.$$

Then, by (2.112) and strict monotonicity of $\widetilde{g}$ we infer

$$\begin{aligned}\mathscr{L}^n\Big(\Big\{\frac{|v_j|}{\Gamma} > 2 - \frac{1}{j}\Big\}\Big) &= \mathscr{L}^n\Big(\Big\{\widetilde{g}\Big(\frac{|v_j|}{\Gamma}\Big) > \widetilde{g}\Big(2 - \frac{1}{j}\Big)\Big\}\Big)\\ &\leqslant \frac{1}{\widetilde{g}\Big(2-\frac{1}{j}\Big)}\int_\Omega g\Big(\frac{v_j}{\Gamma}\Big)\,\mathrm{d}x \leqslant \frac{1}{\widetilde{g}\Big(2-\frac{1}{j}\Big)} \overset{(2.107)}{<} \omega_n\Big(\frac{m}{4C_{\mathtt{opt}}A_j j^4}\Big)^{n(n+1)},\end{aligned}$$

where $\omega_n := \mathscr{L}^n(\mathrm{B}(0,1))$, the ultimate inequality being valid by construction of g. The inequality just proved now implies that the set $\{|v_j|/\Gamma > 2 - \frac{1}{j}\}$ cannot contain any ball of radius $(m/(4C_{\mathtt{opt}}A_j j^4))^{n+1}$. Therefore, for any $x \in \Omega$ with $\frac{|v_j(x)|}{\Gamma} > 2 - \frac{1}{j}$, there exists $y \in \Omega$ with $|x-y| \leqslant (m/(4C_{\mathtt{opt}}A_j j^4))^{n+1}$ and $\frac{|v_j(y)|}{\Gamma} \leqslant 2 - \frac{1}{j}$. In conclusion, as $v_j \in \mathrm{C}(\overline{\Omega};\mathbb{R}^n)$,

$$\frac{|v_j(x)|}{\Gamma} \leqslant \frac{|v_j(y)|}{\Gamma} + \frac{4C_{\mathtt{opt}}A_j j^2}{(1+M+\|\mathfrak{I}\|)m}|x-y|^{\frac{1}{n+1}} \leqslant 2 - \frac{1}{j} + \frac{1}{j^2} < 2.$$

The important upshot of the previous estimation is that $\sup_K |v_j| < 2\Gamma$ for *any* $K \Subset \Omega$. Summarising, we may record that

$$(2.113)\quad \begin{aligned} &\int_\Omega |\varepsilon(v_j)|\,\mathrm{d}x + \frac{1}{A_j j^2}\int_\Omega (1+|\varepsilon(v_j)|^2)^{\frac{n+1}{2}}\,\mathrm{d}x \\ &\qquad\leqslant \frac{1}{\min\{c_1,\frac{1}{2}\}}\Big(\inf F[\mathscr{D}_{u_0}] + \gamma\mathscr{L}^n(\Omega) + \frac{2}{j^2}\Big) \quad \text{for all } j \geqslant j_0, \\ &\sup_{j\in\mathbb{N}} |v_j| \leqslant 2\Gamma, \\ &\sup_K |v_j| < 2\Gamma \quad \text{for all } j \geqslant j_0 \text{ and all } K \Subset \Omega. \end{aligned}$$

Based on these preparations, we may now turn to the perturbed Euler-Lagrange equations satisfied by v_j. Let $\varphi \in \mathrm{W}^{1,n+1}_c(\Omega;\mathbb{R}^n)$ be arbitrary. Then, since $\varphi \in \mathrm{C}_c(\Omega;\mathbb{R}^n)$, the important conclusion of $(2.113)_3$ is that for $\theta > 0$ sufficiently small there holds $|v_j + \theta\varphi| < 6m$, too. Hence there holds $F_j[v_j + \theta\varphi] < \infty$, and since g is differentiable on $\mathbb{B}(0,2)$, the same argument that led to (2.89) now eventually yields

$$(2.114)\qquad \left|\int_\Omega \langle f_j'(\varepsilon(v_j)),\varepsilon(\varphi)\rangle + \Big\langle g'\Big(\frac{v_j}{\Gamma}\Big),\frac{\varphi}{\Gamma}\Big\rangle\,\mathrm{d}x\right| \leqslant \frac{1}{j}\|\varphi\|_{\mathrm{W}^{-2,1}(\Omega;\mathbb{R}^n)}$$

for all $\varphi \in \mathrm{W}^{1,n+1}_c(\Omega;\mathbb{R}^n)$. We conclude this section with various remarks.

Remark 2.67 ($\mathrm{W}^{1,2}$- vs. $\mathrm{W}^{1,n+1}$-regularisations)**.** It is natural to inquire whether it is possible to employ a $\mathrm{W}^{1,2}$-regularisation (as is done in case 1) for case 2, too. Whereas the incorporation of the lower order term in (2.108) would yield – when A_j and accordingly f_j are taken to be the same as in (2.84) – an Ekeland competitor $v_j \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^n) \cap \mathrm{L}^\infty_{\leqslant 2\Gamma}(\Omega;\mathbb{R}^n)$, v_j would not need to be continuous and hence $\max_K |v_j| = 2\Gamma$ might happen for some $K \Subset \Omega$. In this situation, the accordingly modified perturbed Euler-Lagrange equation (2.114) could not be derived in the requisite form. Indeed, $F_j[v_j + t\varphi] = \infty$ then could possibly happen even for some non-trivial $\varphi \in \mathrm{C}^\infty_c(\Omega;\mathbb{R}^n)$, howsoever small $|t|$ might be. Moreover, note that even if $\max_K |v_j| < 2\Gamma$ held on all $K \Subset \Omega$, the failure of the embedding $\mathrm{W}^{1,2}$ into L^∞ (recall that $n \geqslant 2$) would give rise to a similar obstruction.

Remark 2.68 (On bounded minimising sequences)**.** In our setting, the specific minimising sequence as chosen at the very beginning of the approximation schemes for the L^∞-constrained case can be *a priori taken to belong to* $\mathrm{L}^\infty_{\leqslant m}(\Omega;\mathbb{R}^n)$. In our setting, we may put

$$\widetilde{w} := \begin{cases} w & \text{if } |w| \leqslant m, \\ \frac{w}{|w|}m & \text{if } |w| > m \end{cases}$$

which then satisfies $|\nabla\widetilde{w}| \leqslant |\nabla w|$.

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